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S.3323. *Proposed by Dumitru Bătineţu-Giurgiu, and Claudia Nănuţi, Romania.* If $a, b, c, x, y > 0$, then prove that

$$\left(\frac{a\cdot\sqrt{a^3}}{bx+cy}+\frac{b\cdot\sqrt{b^3}}{cx+ay}+\frac{c\cdot\sqrt{c^3}}{ax+by}\right)\cdot\left(\frac{\sqrt[3]{a^2}}{bx+cy}+\frac{\sqrt[3]{b^2}}{cx+ay}+\frac{\sqrt[3]{c^2}}{ax+by}\right)\geq\frac{9}{(x+y)^2}$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Expressing the powers as rational exponents ($a\sqrt[3]{a}=a^{4/3}$ and $\sqrt[3]{a^2}=a^{2/3}$), we apply the Cauchy-Schwarz inequality

$$\left(\sum_{\text{cyc}}\frac{a^{4/3}}{bx+cy}\right)\left(\sum_{\text{cyc}}\frac{a^{2/3}}{bx+cy}\right)\geq\left(\sum_{\text{cyc}}\sqrt{\frac{a^{4/3}}{bx+cy}}\cdot\sqrt{\frac{a^{2/3}}{bx+cy}}\right)^2$$

Simplifying the terms inside the square yields

$$\sqrt{\frac{a^{4/3}}{bx+cy}}\cdot\sqrt{\frac{a^{2/3}}{bx+cy}}=\frac{\sqrt{a^{4/3}\cdot a^{2/3}}}{bx+cy}=\frac{\sqrt{a^2}}{bx+cy}=\frac{a}{bx+cy}$$

Therefore,

$$\left(\sum_{\text{cyc}}\frac{a\sqrt[3]{a}}{bx+cy}\right)\left(\sum_{\text{cyc}}\frac{\sqrt[3]{a^2}}{bx+cy}\right)\geq\left(\sum_{\text{cyc}}\frac{a}{bx+cy}\right)^2$$

Rewriting each term of the sum $\displaystyle\sum_{\text{cyc}}\frac{a}{bx+cy}$ as $\displaystyle\frac{a^2}{abx+acy}$, we get

$$\sum_{\text{cyc}}\frac{a}{bx+cy}=\frac{a^2}{abx+acy}+\frac{b^2}{bcx+bay}+\frac{c^2}{cax+cby}$$

Applying Cauchy's inequality, we have

$$\sum_{\text{cyc}}\frac{a}{bx+cy}\geq\frac{(a+b+c)^2}{(ab+bc+ca)x+(ac+ba+cb)y}=\frac{(a+b+c)^2}{(ab+bc+ca)(x+y)}$$

Using the well-known inequality $(a+b+c)^2\geq 3(ab+bc+ca)$, we obtain

$$\sum_{\text{cyc}}\frac{a}{bx+cy}\geq\frac{3(ab+bc+ca)}{(ab+bc+ca)(x+y)}=\frac{3}{x+y}$$

Squaring both sides yields

$$\left(\sum_{\text{cyc}}\frac{a}{bx+cy}\right)^2\geq\frac{9}{(x+y)^2}$$

Finally, combining the preceding, we have

$$\left(\sum_{\text{cyc}} \frac{a\sqrt[3]{a}}{bx+cy}\right)\left(\sum_{\text{cyc}} \frac{\sqrt[3]{a^2}}{bx+cy}\right) \geq \left(\sum_{\text{cyc}} \frac{a}{bx+cy}\right)^2 \geq \frac{9}{(x+y)^2}$$

Equality holds if and only if $a = b = c$, for any $x, y > 0$.