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S.3314. *Proposed by Dumitru Bătineţu-Giurgiu, and Claudia Nănuţi, Romania.* Let $a, b > 0$, $n \in \mathbb{N}^*$, $n \geq 2$, and $t_k, u_k > 0$ for all $k \in \{1, 2, \ldots, n\}$. Prove that

$$\sum_{k=1}^{n} \left(a \cdot \frac{t_k}{u_k} + b \cdot \frac{u_k}{t_k}\right)^3 + \sum_{k=1}^{n} \left(a \cdot \frac{u_k}{t_k} + b \cdot \frac{t_k}{u_k}\right)^3 \geq 2n(a+b)^3$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. For each $k \in \{1, 2, \ldots, n\}$, define $x_k = \dfrac{t_k}{u_k} > 0$. The inequality simplifies to

$$\sum_{k=1}^{n} \left[\left(ax_k + \frac{b}{x_k}\right)^3 + \left(bx_k + \frac{a}{x_k}\right)^3\right] \geq 2n(a+b)^3$$

Next, we analyze a single term for index k. For a fixed index k, let

$$A_k = ax_k + \frac{b}{x_k}, \quad B_k = bx_k + \frac{a}{x_k}$$

Adding A_k and B_k yields

$$A_k + B_k = (a+b)x_k + \frac{a+b}{x_k} = (a+b)\left(x_k + \frac{1}{x_k}\right)$$

By the convexity of $f(t) = t^3$ for $t > 0$), we have

$$\frac{A_k^3 + B_k^3}{2} \geq \left(\frac{A_k + B_k}{2}\right)^3 \Rightarrow A_k^3 + B_k^3 \geq \frac{(A_k + B_k)^3}{4}$$

Substituting $A_k + B_k$ yields

$$A_k^3 + B_k^3 \geq \frac{(a+b)^3 \left(x_k + \frac{1}{x_k}\right)^3}{4}$$

By the AM-GM Inequality, for any $x_k > 0$, we have

$$x_k + \frac{1}{x_k} \geq 2\sqrt{x_k \cdot \frac{1}{x_k}} = 2 \Rightarrow \left(x_k + \frac{1}{x_k}\right)^3 \geq 8$$

Substituting this lower bound back into the preceding, yields

$$A_k^3 + B_k^3 \geq \frac{(a+b)^3 \cdot 8}{4} = 2(a+b)^3$$

Finally, summing $A_k^3 + B_k^3 \geq 2(a+b)^3$ for all $k = 1, 2, \ldots, n$, we conclude that

$$\sum_{k=1}^{n} \left[\left(a \cdot \frac{t_k}{u_k} + b \cdot \frac{u_k}{t_k}\right)^3 + \left(a \cdot \frac{u_k}{t_k} + b \cdot \frac{t_k}{u_k}\right)^3\right] \geq \sum_{k=1}^{n} 2(a+b)^3 = 2n(a+b)^3$$

Equality holds if and only if $t_k = u_k$ for all $k \in \{1, 2, \ldots, n\}$.