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S.3311. *Proposed by Dumitru Bătineţu-Giurgiu, and Claudia Nănuţi, Romania.* If $x, y > 0$ and $x + y = 4$, then in any triangle ABC with area F, and medians m_a, m_b, m_c,the following inequality holds

$$m_a^x + m_b^x + m_c^x + m_a^y + m_b^y + m_c^y \geq 6\sqrt{3}\,F$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. By the AM-GM inequality, for any positive real numbers u and v it holds $u + v \geq 2\sqrt{uv}$. Applying this to the terms m_a^x and m_a^y, we get

$$m_a^x + m_a^y \geq 2\sqrt{m_a^x \cdot m_a^y} = 2\sqrt{m_a^{x+y}}$$

Since $x + y = 4$, then we have

$$m_a^x + m_a^y \geq 2\sqrt{m_a^4} = 2m_a^2$$

Similarly, for the medians m_b and m_c, we have

$$m_b^x + m_b^y \geq 2m_b^2$$
$$m_c^x + m_c^y \geq 2m_c^2$$

Adding up these three inequalities gives

$$m_a^x + m_b^x + m_c^x + m_a^y + m_b^y + m_c^y \geq 2(m_a^2 + m_b^2 + m_c^2)$$

By Apollonius's Theorem, the square of the median m_a is given by

$$m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4}$$

Summing the squares of all three medians yields

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$$

Substituting this into our preceding bound yields

$$2(m_a^2 + m_b^2 + m_c^2) = 2 \cdot \frac{3}{4}(a^2 + b^2 + c^2) = \frac{3}{2}(a^2 + b^2 + c^2)$$

Weitzenböck's Inequality, for any triangle with sides a, b, c and area F states:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}\,F$$

Substituting this into our lower bound yields

$$\frac{3}{2}(a^2 + b^2 + c^2) \geq \frac{3}{2}\left(4\sqrt{3}\,F\right) = 6\sqrt{3}\,F$$

Combining all preceding steps gives

$$m_a^x + m_b^x + m_c^x + m_a^y + m_b^y + m_c^y \geq 2(m_a^2 + m_b^2 + m_c^2) \geq 6\sqrt{3}\,F$$

Equality holds if and only if the triangle ABC is equilateral ($a = b = c$) and $x = y = 2$.