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S.3310. *Proposed by Dumitru Bătineţu-Giurgiu, and Claudia Nănuţi, Romania.* If $a, b, c > 0$, then prove that

$$\left(\frac{a\sqrt[3]{a}}{b+c}+\frac{b\sqrt[3]{b}}{c+a}+\frac{c\sqrt[3]{c}}{a+b}\right)\left(\frac{\sqrt[3]{a^2}}{b+c}+\frac{\sqrt[3]{b^2}}{c+a}+\frac{\sqrt[3]{c^2}}{a+b}\right)\geq\frac{9}{4}$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Rewriting the expression using fractional exponents, we have to show that

$$\left(\frac{a^{4/3}}{b+c}+\frac{b^{4/3}}{c+a}+\frac{c^{4/3}}{a+b}\right)\left(\frac{a^{2/3}}{b+c}+\frac{b^{2/3}}{c+a}+\frac{c^{2/3}}{a+b}\right)\geq\frac{9}{4}$$

Next, we will apply Cauchy-Schwarz Inequality:

$$(x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2)\geq(x_1y_1+x_2y_2+x_3y_3)^2$$

with

$$x_1=\sqrt{\frac{a^{4/3}}{b+c}},\quad x_2=\sqrt{\frac{b^{4/3}}{c+a}},\quad x_3=\sqrt{\frac{c^{4/3}}{a+b}}$$

$$y_1=\sqrt{\frac{a^{2/3}}{b+c}},\quad y_2=\sqrt{\frac{b^{2/3}}{c+a}},\quad y_3=\sqrt{\frac{c^{2/3}}{a+b}}$$

Indeed, multiplying the corresponding terms yields

$$x_1y_1=\sqrt{\frac{a^{4/3}\cdot a^{2/3}}{(b+c)^2}}=\sqrt{\frac{a^2}{(b+c)^2}}=\frac{a}{b+c}$$

Similarly, $x_2y_2=\dfrac{b}{c+a}$ and $x_3y_3=\dfrac{c}{a+b}$. Then, on account of CBS, we have

$$\left(\frac{a^{4/3}}{b+c}+\frac{b^{4/3}}{c+a}+\frac{c^{4/3}}{a+b}\right)\left(\frac{a^{2/3}}{b+c}+\frac{b^{2/3}}{c+a}+\frac{c^{2/3}}{a+b}\right)\geq\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2$$

Nesbitt's Inequality, for any positive real numbers a, b, c states that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}$$

Squaring both sides gives

$$\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2\geq\left(\frac{3}{2}\right)^2=\frac{9}{4}$$

Finally, combining the preceding results yields

$$\left(\frac{a\sqrt[3]{a}}{b+c}+\frac{b\sqrt[3]{b}}{c+a}+\frac{c\sqrt[3]{c}}{a+b}\right)\left(\frac{\sqrt[3]{a^2}}{b+c}+\frac{\sqrt[3]{b^2}}{c+a}+\frac{\sqrt[3]{c^2}}{a+b}\right)\geq\frac{9}{4}$$

Equality holds if and only if $a=b=c$.