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S.3304. *Proposed by Dumitru Bătineţu-Giurgiu, and Claudia Nănuţi, Romania.* If $a, b > 0$ and $x \in \mathbb{R}$, then prove that

$$(a\sin^2 x + b)^3 + (a\cos^2 x + b)^3 \geq \frac{1}{4}(a + 2b)^3$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Let $u = a\sin^2 x + b$ and $v = a\cos^2 x + b$. Since $a, b > 0$, both u and v are positive real numbers. Using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, we have

$$u + v = (a\sin^2 x + b) + (a\cos^2 x + b) = a(\sin^2 x + \cos^2 x) + 2b = a + 2b$$

Consider the expression $u^3 + v^3 - \frac{1}{4}(u+v)^3$. We have

$$\begin{aligned} u^3 + v^3 - \frac{1}{4}(u+v)^3 &= (u+v)\left(u^2 - uv + v^2\right) - \frac{1}{4}(u+v)(u+v)^2 \\ &= (u+v)\left[u^2 - uv + v^2 - \frac{1}{4}(u^2 + 2uv + v^2)\right] \\ &= (u+v)\left(\frac{3}{4}u^2 - \frac{3}{2}uv + \frac{3}{4}v^2\right) \\ &= \frac{3}{4}(u+v)(u-v)^2 \end{aligned}$$

Since $u > 0$ and $v > 0$, we have $u + v > 0$ and $(u - v)^2 \geq 0$. Therefore,

$$u^3 + v^3 - \frac{1}{4}(u+v)^3 = \frac{3}{4}(u+v)(u-v)^2 \geq 0$$

$$\Rightarrow u^3 + v^3 \geq \frac{1}{4}(u+v)^3$$

Substituting $u = a\sin^2 x + b$, $v = a\cos^2 x + b$, and $u + v = a + 2b$ yields

$$(a\sin^2 x + b)^3 + (a\cos^2 x + b)^3 \geq \frac{1}{4}(a + 2b)^3$$

Equality holds if and only if $u = v$, which means $\sin^2 x = \cos^2 x \Rightarrow \tan^2 x = 1$. This occurs when $x = \dfrac{\pi}{4} + \dfrac{k\pi}{2}$ for any $k \in \mathbb{Z}$.