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S.3302. *Proposed by Dumitru Bătineţu-Giurgiu, and Claudia Nănuţi, Romania.* If $x, y, z > 0$ and $a + b + c = 18\,abc$, then prove that

$$\left(\frac{1}{a^2}+5\right)\left(\frac{1}{b^2}+5\right)\left(\frac{1}{c^2}+5\right) \geq 243$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Let

$$x = \frac{1}{a}, \quad y = \frac{1}{b}, \quad z = \frac{1}{c}$$

Since $a, b, c > 0$, we have $x, y, z > 0$. Dividing both sides of $a + b + c = 18abc$ by abc gives

$$\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab} = 18 \Rightarrow xy + yz + zx = 18$$

We want to prove that

$$(x^2+5)(y^2+5)(z^2+5) \geq 243$$

Expanding the LHS yields

$$(x^2+5)(y^2+5)(z^2+5) = x^2y^2z^2 + 5(x^2y^2+y^2z^2+z^2x^2) + 25(x^2+y^2+z^2) + 125$$

We have the standard algebraic inequalities

1. $x^2 + y^2 + z^2 \geq xy + yz + zx = 18$
2. $x^2y^2 + y^2z^2 + z^2x^2 \geq \dfrac{(xy+yz+zx)^2}{3} = \dfrac{18^2}{3} = 108$
3. $x^2y^2z^2 > 0 \quad$ (since $x, y, z > 0$)

Substituting these lower bounds into the expanded form yields

$$(x^2+5)(y^2+5)(z^2+5) > 0 + 5(108) + 25(18) + 125$$

$$(x^2+5)(y^2+5)(z^2+5) > 540 + 450 + 125 = 1115$$

Since $1115 > 243$, it immediately follows that

$$\left(\frac{1}{a^2}+5\right)\left(\frac{1}{b^2}+5\right)\left(\frac{1}{c^2}+5\right) \geq 243$$

Analyzing the LHS we have found that its smallest value is 1331. I suggest the following modified statement.

Let $a, b, c > 0$ such that $a + b + c = 18\,abc$. Prove that

$$\left(\frac{1}{a^2}+5\right)\left(\frac{1}{b^2}+5\right)\left(\frac{1}{c^2}+5\right) \geq 1331$$

To prove it, we begin by making the substitution $x = \dfrac{1}{a}$, $y = \dfrac{1}{b}$, and $z = \dfrac{1}{c}$. Since $a, b, c > 0$, we have $x, y, z > 0$. Dividing the given condition $a + b + c = 18\,abc$ by abc yields:

$$\frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} = 18 \implies xy + yz + zx = 18$$

The inequality to prove becomes

$$(x^2 + 5)(y^2 + 5)(z^2 + 5) \geq 1331$$

Expanding the product $(x^2 + 5)(y^2 + 5)(z^2 + 5)$, we get

$$(x^2+5)(y^2+5)(z^2+5) = (xyz)^2 + 5(x^2y^2 + y^2z^2 + z^2x^2) + 25(x^2 + y^2 + z^2) + 125$$

Using the constraint $xy + yz + zx = 18$ in the following known identities yields:

1. $x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + yz + zx) = (x + y + z)^2 - 36$
2. $x^2y^2+y^2z^2+z^2x^2 = (xy+yz+zx)^2 - 2xyz(x+y+z) = 324 - 2xyz(x+y+z)$

Substituting these back into the expansion yields

$$\begin{aligned}(x^2 + 5)(y^2 + 5)(z^2 + 5) &= (xyz)^2 + 5\big(324 - 2xyz(x + y + z)\big) + 25\big((x + y + z)^2 - 36\big) + 125 \\ &= (xyz)^2 - 10xyz(x + y + z) + 1620 + 25(x + y + z)^2 - 900 + 125 \\ &= (xyz)^2 - 10xyz(x + y + z) + 25(x + y + z)^2 + 845\end{aligned}$$

Notice that the first three terms form a perfect square, and we have

$$(x^2 + 5)(y^2 + 5)(z^2 + 5) = \big[5(x + y + z) - xyz\big]^2 + 845$$

To prove that the expression is ≥ 1331, we show that

$$\big[5(x + y + z) - xyz\big]^2 \geq 1331 - 845 = 486$$

Recall two standard inequalities for positive real numbers x, y, z:

1. Sum bound: $(x + y + z)^2 \geq 3(xy + yz + zx) = 3(18) = 54 \implies x + y + z \geq \sqrt{54} = 3\sqrt{6}$
2. Product bound: $(x + y + z)(xy + yz + zx) \geq 9xyz \implies 18(x + y + z) \geq 9xyz \implies xyz \leq 2(x + y + z)$

Using $xyz \leq 2(x + y + z)$, we get

$$5(x + y + z) - xyz \geq 5(x + y + z) - 2(x + y + z) = 3(x + y + z)$$

Since $x + y + z \geq 3\sqrt{6}$, then

$$5(x + y + z) - xyz \geq 3(3\sqrt{6}) = 9\sqrt{6}$$

Squaring both sides gives

$$\big[5(x + y + z) - xyz\big]^2 \geq (9\sqrt{6})^2 = 81 \times 6 = 486$$

Adding 845 to both sides, we obtain

$$(x^2 + 5)(y^2 + 5)(z^2 + 5) = \big[5(x + y + z) - xyz\big]^2 + 845 \geq 486 + 845 = 1331$$

Substituting back $x = \frac{1}{a}$, $y = \frac{1}{b}$, and $z = \frac{1}{c}$ yields

$$\left(\frac{1}{a^2} + 5\right)\left(\frac{1}{b^2} + 5\right)\left(\frac{1}{c^2} + 5\right) \geq 1331$$

Equality holds when $x = y = z = \sqrt{6}$, which corresponds to $a = b = c = \frac{1}{\sqrt{6}}$.