ROMANIAN MATHEMATICAL MAGAZINE CHALLENGES - II

DANIEL SITARU - ROMANIA

01. If $A, B \in M_2(\mathbb{R}); A^{2027} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}; B^{2025} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ then find:

$$M = \det A + \det B$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$A^{2027} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \Rightarrow \det(A^{2027}) = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}$$

$$(\det A)^{2027} = 2 \cdot 3 - 1 \cdot 5 \Rightarrow (\det A)^{2027} = 1 \Rightarrow \det A = 1$$

$$B^{2025} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \Rightarrow \det(B^{2025}) = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \Rightarrow$$

$$\Rightarrow (\det B)^{2025} = 1 \cdot 3 - 2 \cdot 2 \Rightarrow (\det B)^{2025} = -1 \Rightarrow \det B = -1$$

$$M = \det A + \det B = 1 - 1 = 0$$

02. If $A, B \in M_2(\mathbb{R}); A^2 - AB = 8I_2; B^2 - BA = I_2$ then find:

$$M = \det A + \det B$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$A^2 - AB = 8I_2 \Rightarrow A(A - B) = 8I_2 \Rightarrow \det(A(A - B)) = \det(8I_2)$$

$$\det A \cdot \det(A - B) = 64 \neq 0 \Rightarrow (\exists)A^{-1}$$

$$A^2 - AB = 8I_2 \Rightarrow A^{-1}(A^2 - AB) = 8A^{-1}I_2$$

$$A^{-1}A^2 - A^{-1}AB = 8A^{-1} \Rightarrow A - B = 8A^{-1} \Rightarrow A = B + 8A^{-1}$$

$$B^2 - BA = I_2 \Rightarrow B^2 - B(B + 8A^{-1}) = I_2$$

$$B^2 - B^2 - 8BA^{-1} = I_2 \Rightarrow -8BA^{-1} = I_2 \Rightarrow 8BA^{-1}A = I_2 \cdot A$$

$$\Rightarrow -8BI_2 = A \Rightarrow A = -8B$$

$$B^2 - BA = I_2 \Rightarrow B^2 - B(-8B) = I_2 \Rightarrow 9B^2 = I_2$$

$$B^2 = \frac{1}{9}I_2 \Rightarrow \det(B^2) = \det\left(\frac{1}{9}I_2\right) \Rightarrow (\det B)^2 = \frac{1}{81} \Rightarrow \det B = \frac{1}{9}$$

$$A = -8B \Rightarrow \det A = \det(-8B) \Rightarrow \det A = (-8)^2 \det B$$

$$\det A = 64 \cdot \frac{1}{9} = \frac{64}{9}$$

$$M = \det A + \det B = \frac{64}{9} + \frac{1}{9} = \frac{65}{9}$$

03. If $a, b, c > 0; abc = 1$ then:

$$\frac{(a+2)^2}{a^3} + \frac{(b+2)^2}{b^3} + \frac{(c+2)^2}{c^3} \geq 27$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{(a+2)^2}{a^3} + \frac{(b+2)^2}{b^3} + \frac{(c+2)^2}{c^3} \overset{\text{AM-GM}}{\geq}$$

$$\geq 3\sqrt[3]{\frac{((a+2)(b+2)(c+2))^2}{(abc)^2}} = 3\sqrt[3]{((a+2)(b+2)(c+2))^2}$$

Remains to prove:

$$3\sqrt[3]{((a+2)(b+2)(c+2))^2} \geq 27 \Leftrightarrow \sqrt[3]{((a+2)(b+2)(c+2))^2} \geq 9$$

$$((a+2)(b+2)(c+2))^2 \geq 9^3 (a+2)(b+2)(c+2) \geq 3^3$$

$$abc + 2(ab+bc+ca) + 4(a+b+c) + 8 \geq 27$$

$$2(ab+bc+ca) + 4(a+b+c) \geq 18$$

$$ab+bc+ca+2(a+b+c) \geq 9 \text{ (to prove)}$$

$$ab+bc+ca+2(a+b+c) \overset{\text{AM-GM}}{\geq} 9\sqrt[9]{ab \cdot bc \cdot ca \cdot a^2 \cdot b^2 \cdot c^2} = 9\sqrt[3]{(abc)^4} = 9 \cdot 1 = 9$$

Equality holds for $a = b = c = 1$.

04. In acute ΔABC the following relationship holds:

$$\frac{abc}{HA \cdot HB \cdot HC} \geq 3\sqrt{3}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{abc}{HA \cdot HB \cdot HC} \geq 3\sqrt{3} \Leftrightarrow \frac{2R\sin A \cdot 2R\sin B \cdot 2R\sin C}{2R\cos A \cdot 2R\cos B \cdot 2R\cos C} \geq 3\sqrt{3}$$

$$\frac{\sin A}{\cos A} \cdot \frac{\sin B}{\cos B} \cdot \frac{\sin C}{\cos C} \geq 3\sqrt{3}$$

$$\tan A \cdot \tan B \cdot \tan C \geq 3\sqrt{3}$$

$$\tan A + \tan B + \tan C \geq 3\sqrt{3} \text{ (to prove)}$$

$$\tan A + \tan B + \tan C \overset{\text{JENSEN}}{\geq} 3\tan\left(\frac{A+B+C}{3}\right) = 3\tan\left(\frac{\pi}{3}\right) = 3\sqrt{3}$$

Equality holds for $A = B = C$.

05. In acute ABC the following relationship holds:

$$\frac{HA}{a\cos A} + \frac{HB}{b\cos B} + \frac{HC}{c\cos C} \geq 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{HA}{a\cos A}+\frac{HB}{b\cos B}+\frac{HC}{c\cos C}=\sum_{cyc}\frac{HA}{a\cos A}=$$

$$=\sum_{cyc}\frac{2R\cos A}{a\cos A}=2R\sum_{cyc}\frac{1}{a}=\sum_{cyc}\frac{2R}{2R\sin A}=$$

$$=\sum_{cyc}\frac{1}{\sin A}\overset{\text{JENSEN}}{\geq}3\cdot\frac{1}{\sin\left(\frac{A+B+C}{3}\right)}=\frac{3}{\sin\frac{\pi}{3}}=\frac{3}{\frac{\sqrt{3}}{2}}=\frac{6}{\sqrt{3}}=\frac{6\sqrt{3}}{3}=2\sqrt{3}$$

Equality holds for $a=b=c$.

06. In acute ΔABC the following relationship holds:

$$\frac{HA}{a}+\frac{HB}{b}+\frac{HC}{c}\geq\sqrt{3}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{HA}{a}+\frac{HB}{b}+\frac{HC}{c}=\sum_{cyc}\frac{HA}{a}=\sum_{cyc}\frac{2R\cos A}{2R\sin A}=\sum_{cyc}\frac{b^2+c^2-a^2}{2bc\sin A}=\sum_{cyc}\frac{b^2+c^2-a^2}{4F}=$$

$$=\frac{1}{4F}\left(\sum_{cyc}b^2+\sum_{cyc}c^2-\sum_{cyc}a^2\right)=\frac{1}{4F}\sum_{cyc}a^2=$$

$$=\frac{1}{4F}(a^2+b^2+c^2)\overset{\text{IONESCU-WEITZENBOCK}}{\geq}\frac{1}{4F}\cdot 4\sqrt{3}F=\sqrt{3}$$

Equality holds for $a=b=c$.

07. If $A,B\in M_n(\mathbb{R}); AB+9A+4B=O_n$ then:

$$det^2(A+4I_n)+det^2(B+9I_n)\geq 72$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$(A+4I_n)(B+9I_n)=AB+9A+4B+36I_n=36I_n$$

$$\det((A+4I_n)(B+9I_n))=\det(36I_n)$$

$$\det(A+4I_n)\cdot\det(B+9I_n)=36^2$$

$$|\det(A+4I_n)|\cdot|\det(B+9I_n)|=36^2$$

$$det^2(A+4I_n)+det^2(B+9I_n)\overset{\text{AM-GM}}{\geq}2\sqrt{|\det(A+4I_n)|\cdot\det(B+9I_n)|}=$$

$$=2\sqrt{|\det(A+4I_n)|\cdot|\det(B+9I_n)|}=2\sqrt{36^2}=72$$

Equality holds for: $|\det(A+4I_n)|=|\det(B+9I_n)|$.

08. If $A,B\in M_2(\mathbb{R}): A^2B=\begin{pmatrix}-1 & 2\\ 3 & 5\end{pmatrix}; B^2A=\begin{pmatrix}1 & 2\\ 2 & 6\end{pmatrix}$ then find:
$M=\det A+\det B$.

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$A^2B = \begin{pmatrix} -1 & 2 \\ 3 & 5 \end{pmatrix} \Rightarrow \det(A^2B) = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} \Rightarrow$$

$$(\det A)^2 \cdot \det B = -5 + 6 \Rightarrow \det B = \frac{1}{(\det A)^2}$$

$$B^2A = \begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix} \Rightarrow \det(B^2A) = \begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} \Rightarrow (\det B)^2 \cdot \det A = 6 \cdot 1 - 2 \cdot 2 = 6 - 4 = 2$$

$$\frac{1}{(\det A)^4} \cdot \det A = 2 \Rightarrow \frac{1}{(\det A)^3} = 2 \Rightarrow \det A = \frac{1}{\sqrt[3]{2}}$$

$$\det B = \frac{1}{(\det A)^2} = \sqrt[3]{4}$$

$$M = \det A + \det B = \frac{1}{\sqrt[3]{2}} + \sqrt[3]{4} = \frac{3}{\sqrt[3]{2}}$$

09. Solve for real numbers:

$$\sqrt{1+x} + \sqrt{1-x} = 2(1-x^2)$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\text{Denote: } \sqrt{1+x} = a > 0; \sqrt{1-x} = b > 0 \Rightarrow P = ab > 0; S = a + b$$

$$\begin{matrix} \sqrt{1+x} = a \Rightarrow 1 + x = a^2 \\ \sqrt{1-x} = b \Rightarrow 1 - x = b^2 \end{matrix} \Rightarrow a^2 + b^2 = 2 \Rightarrow S^2 - 2P = 2$$

$$\sqrt{1+x} + \sqrt{1-x} = 2(1-x)(1+x) \Rightarrow a + b = 2a^2b^2$$

$$S = 2P^2 \Rightarrow (2P^2)^2 - 2P = 2$$

$$4P^4 - 2P - 2 = 0 \Rightarrow 2P^4 - P - 1 = 0$$

$$2P^4 - 2P + P - 1 = 0 \Rightarrow 2P(P^3 - 1) + (P - 1) = 0$$

$$2P(P-1)(P^2 + P + 1) + (P - 1) = 0$$

$$(P-1)(2P^3 + 2P^2 + 2P + 1) = 0$$

$$P > 0 \Rightarrow 2P^3 + 2P^2 + 2P + 1 \neq 0$$

$$P - 1 = 0 \Rightarrow P = 1 \Rightarrow S = 2 \Rightarrow a = b = 1$$

$$\sqrt{1+x} = 1 \Rightarrow 1 + x = 1 \Rightarrow x = 0$$

10. If $a, b, c > 0, ab + bc + ca = 1$ then:

$$\frac{(ab+1)^2}{a^2 + 4ab + b^2} + \frac{(bc+1)^2}{b^2 + 4bc + c^2} + \frac{(ca+1)^2}{c^2 + 4ca + a^2} \geq \frac{8}{3}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$a^2+b^2+c^2 \geq ab+bc+ca=1$$

(1) $$a^2+b^2+c^2 \geq 1$$

$$\frac{(ab+1)^2}{a^2+4ab+b^2}+\frac{(bc+1)^2}{b^2+4bc+c^2}+\frac{(ca+1)^2}{c^2+4ca+a^2}=\sum_{cyc}\frac{(ab+1)^2}{a^2+4ab+b^2}\overset{\text{BERGSTRÖM}}{\geq}$$

$$\geq \frac{(ab+1+bc+1+ca+1)^2}{2(a^2+b^2+c^2)+4(ab+bc+ca)}=\frac{(1+3)^2}{2(a^2+b^2+c^2)+4}=$$

$$=\frac{16}{2(a^2+b^2+c^2)+4}\geq\frac{8}{a^2+b^2+c^2+2}\overset{(1)}{\geq}\frac{8}{1+2}=\frac{8}{3}$$

Equality holds for $a=b=c=\frac{\sqrt{3}}{8}$.

11. In acute ΔABC the following relationship holds:

$$\frac{a}{HA\sin A}+\frac{b}{HB\sin B}+\frac{c}{HC\sin C}\geq 6$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a}{HA\sin A}+\frac{b}{HB\sin B}+\frac{c}{HC\sin C}=\sum_{cyc}\frac{a}{HA\sin A}=$$

$$=\sum_{cyc}\frac{2R\sin A}{2R\cos A\cdot\sin A}=\sum_{cyc}\frac{1}{\cos A}\overset{\text{JENSEN}}{\geq}3\cdot\frac{1}{\cos\left(\frac{A+B+C}{3}\right)}=\frac{3}{\cos\frac{\pi}{3}}=\frac{3}{\frac{1}{2}}=6$$

Equality holds for $a=b=c$.

12. In ΔABC the following relationship holds:

$$r_ar_br_c\geq 27r^3$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$r_ar_br_c=\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}=\frac{F^3}{(s-a)(s-b)(s-c)}=$$

$$=\frac{F^3s}{s(s-a)(s-b)(s-c)}=\frac{F^3s}{F^2}=Fs=rs\cdot s=rs^2\overset{\text{MITRINOVIC}}{\geq}r\cdot(3\sqrt{3}r)^2=r\cdot 27r^2=27r^3$$

Equality holds for $a=b=c$.

13. In acute ΔABC the following relationship holds:

$$\frac{HA\sin A}{a}+\frac{HB\sin B}{b}+\frac{HC\sin C}{c}\leq\frac{3}{2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{HA\sin A}{a}+\frac{HB\sin B}{b}+\frac{HC\sin C}{c}=$$

$$=\sum_{cyc}\frac{HA\sin A}{a}=\sum_{cyc}\frac{2R\cos A\sin A}{2R\sin A}=\sum_{cyc}\cos A\overset{\text{JENSEN}}{\leq}3\cos\left(\frac{A+B+C}{3}\right)=3\cos\frac{\pi}{3}=\frac{3}{2}$$

Equality holds for $a=b=c$.

14. In ΔABC the following relationship holds:

$$\frac{GA^2}{a^2}+\frac{GB^2}{b^2}+\frac{GC^2}{c^2}\geq\frac{3}{2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{GA^2}{a^2}+\frac{GB^2}{b^2}+\frac{GC^2}{c^2}=\sum_{cyc}\frac{GA^2}{a^2}=\frac{2}{3}\sum_{cyc}\frac{m_a^2}{a^2}=$$

$$=\frac{2}{3}\sum_{cyc}\frac{2(b^2+c^2)-a^2}{4a^2}=\frac{1}{6}\sum_{cyc}\left(\frac{2b^2}{a^2}+\frac{2c^2}{a^2}\right)-\frac{1}{6}\sum_{cyc}\frac{a^2}{a^2}=$$

$$=\frac{2}{6}\left(\left(\frac{b^2}{a^2}+\frac{a^2}{b^2}\right)+\left(\frac{c^2}{b^2}+\frac{b^2}{c^2}\right)+\left(\frac{a^2}{c^2}+\frac{c^2}{a^2}\right)\right)-\frac{1}{6}\cdot 3\geq\frac{1}{3}(2+2+2)-\frac{1}{2}=\frac{6}{3}-\frac{1}{2}=2-\frac{1}{2}=\frac{3}{2}$$

Equality holds for $a=b=c$.

15. In acute ΔABC the following relationship holds:

$$\frac{a\cos A}{HA}+\frac{b\cos B}{HB}+\frac{c\cos C}{HC}\leq\frac{3\sqrt{3}}{2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a\cos A}{HA}+\frac{b\cos B}{HB}+\frac{c\cos C}{HC}=\sum_{cyc}\frac{a\cos A}{HA}=$$

$$=\sum_{cyc}\frac{2R\sin A\cdot\cos A}{2R\cos A}=\sum_{cyc}\sin A\overset{\text{JENSEN}}{\leq}3\sin\left(\frac{A+B+C}{3}\right)=3\sin\frac{\pi}{3}=3\cdot\frac{\sqrt{3}}{2}=\frac{3\sqrt{3}}{2}$$

Equality holds for $a=b=c$.

16. Solve for real numbers:

$$\sqrt[3]{1+x}+\sqrt[3]{1-x}=\sqrt[3]{1-x^2}+1$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\left.\begin{array}{l}\sqrt[3]{1+x}=a\Rightarrow 1+x=a^3\\ \sqrt[3]{1-x}=b\Rightarrow 1-x=b^3\end{array}\right\}\Rightarrow a^3+b^3=2$$

Denote: $S=a+b;P=ab$.

$$\sqrt[3]{1+x}+\sqrt[3]{1-x}=\sqrt[3]{(1-x)(1+x)}+1$$
$$a+b=ab+1\Rightarrow S=P+1$$
$$a^3+b^3=2\Rightarrow S^3-3SP=2\Rightarrow(P+1)^3-3P(P+1)=2$$
$$P^3+3P^2+3P+1-3P^2-3P=2$$
$$P^3+1=2\Rightarrow P^3=1\Rightarrow P=1$$
$$S=2;P=1\Rightarrow a=b=1$$
$$\sqrt[3]{1+x}=1\Rightarrow 1+x=1\Rightarrow x=0$$

17. In ΔABC the following relationship holds:

$$GA^2+GB^2+GC^2\leq 3R^2$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$GA^2+GB^2+GC^2=\sum_{cyc}GA^2=\sum_{cyc}\left(\frac{2}{3}m_a\right)^2=\frac{4}{9}\sum_{cyc}m_a^2=$$
$$=\frac{4}{9}\cdot\frac{3}{4}(a^2+b^2+c^2)=\frac{1}{3}(a^2+b^2+c^2)=\frac{1}{3}\cdot 2(s^2-r^2-4Rr)\overset{\text{GERRETSEN}}{\leq}$$
$$\leq\frac{2}{3}(4R^2+3r^2+4Rr-r^2-4Rr)=$$
$$=\frac{2}{3}(4R^2+2r^2)=\frac{2}{3}\left(4R^2+2\cdot\frac{R^2}{4}\right)=\frac{2}{3}\cdot\frac{18R^2}{4}=\frac{18R^2}{6}=3R^2$$

Equality holds for $a=b=c$.

18. In ΔABC the following relationship holds:

$$\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\geq\frac{6r}{R}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}=\sum_{cyc}\frac{h_a}{r_a}=\sum_{cyc}\frac{\frac{2F}{2}}{\frac{F}{s-a}}=$$
$$=2\sum_{cyc}\frac{s-a}{a}=2\left(s\sum_{cyc}\frac{1}{a}-\sum_{cyc}\frac{a}{a}\right)=$$
$$=2\left(s\cdot\frac{ab+bc+ca}{abc}-3\right)=2\left(s\cdot\frac{s^2+r^2+4Rr}{4Rrs}-3\right)=$$
$$=2\left(\frac{s^2+r^2+4Rr-12Rr}{4Rr}\right)\overset{\text{GERRETSEN}}{\geq}$$
$$\geq 2\cdot\frac{16Rr-5r^2+r^2-8Rr}{4Rr}=\frac{8Rr-4r^2}{2Rr}=\frac{4R-2r}{R}\overset{\text{EULER}}{\geq}\frac{4\cdot 2r-2r}{R}=\frac{6r}{R}$$

Equality holds for $a = b = c$.

19. In ΔABC the following relationship holds:

$$\frac{h_a}{r_b} + \frac{h_b}{r_c} + \frac{h_c}{r_a} \geq \frac{6r}{R}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{h_a}{r_b} + \frac{h_b}{r_c} + \frac{h_c}{r_a} = \sum_{cyc} \frac{\frac{2F}{a}}{\frac{F}{s-b}} = 2\sum_{cyc} \frac{s-b}{a} \geq$$

$$\overset{\text{AM-GM}}{\geq} 2 \cdot 3\sqrt[3]{\frac{(s-a)(s-b)(s-c)}{abc}} = \sqrt[3]{\frac{s(s-a)(s-b)(s-c)}{s \cdot 4RF}} =$$

$$= 6\sqrt[3]{\frac{F^2}{4sRF}} = 6\sqrt[3]{\frac{F}{4sR}} = 6\sqrt[3]{\frac{rs}{4sR}} = 6\sqrt[3]{\frac{r}{4R}} \geq \frac{6r}{R} \Leftrightarrow$$

$$\Leftrightarrow \frac{r}{4R} \geq \left(\frac{r}{R}\right)^3 \Leftrightarrow \frac{1}{4} \geq \frac{r^2}{R^2} \Leftrightarrow 4r^2 \Leftrightarrow R \geq 2r$$

Equality holds for $a = b = c$.

20. In ΔABC the following relationship holds:

$$\frac{h_a h_b}{r_a r_b} + \frac{h_b h_c}{r_b r_c} + \frac{h_c h_a}{r_c r_a} \geq \frac{12r^2}{R^2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{h_a h_b}{r_a r_b} + \frac{h_b h_c}{r_b r_c} + \frac{h_c h_a}{r_c r_a} \overset{\text{AM-GM}}{\geq} 3\sqrt[3]{\left(\frac{h_a h_b h_c}{r_a r_b r_c}\right)^2} =$$

$$= 3\sqrt[3]{\left(\frac{\frac{2F}{a} \cdot \frac{2F}{b} \cdot \frac{2F}{c}}{\frac{F}{s-a} \cdot \frac{F}{s-b} \cdot \frac{F}{s-c}}\right)^2} = 3\sqrt[3]{\left(\frac{8(s-a)(s-b)(s-c)}{abc}\right)^2} =$$

$$= 3\sqrt[3]{\left(\frac{8s(s-a)(s-b)(s-c)}{sabc}\right)^2} = 3\sqrt[3]{\left(\frac{8F^2}{s \cdot 4RF}\right)^2} =$$

$$= 3\sqrt[3]{\left(\frac{2F}{Rs}\right)^2} = 3\sqrt[3]{\left(\frac{2rs}{Rs}\right)^2} = 3\sqrt[3]{4\left(\frac{r}{R}\right)^2} \geq 12\left(\frac{r}{R}\right)^2 \Leftrightarrow 27 \cdot 4\left(\frac{r}{R}\right)^2 \geq 12^3 \cdot \left(\frac{r}{R}\right)^6$$

$$3^3 \cdot 4 \geq 3^3 \cdot 4^3 \cdot \left(\frac{r}{R}\right)^4$$

$$\frac{1}{16} \geq \left(\frac{r}{R}\right)^4 \Leftrightarrow \frac{1}{2} \geq \frac{r}{R} \Leftrightarrow R \geq 2r \text{ (Euler)}$$

Equality holds for $a = b = c$.

21. If $a, b > 0; a + b = 2$ then:

$$a^4 + b^4 + 6ab \geq 8$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\text{Denote: } S = a+b; P = ab$$

$$S = a+b = 2; 2 = a+b \overset{\text{AM-GM}}{\geq} 2\sqrt{ab} = 2\sqrt{P}$$

$$2 \geq 2\sqrt{P} \Rightarrow \sqrt{P} \leq 1 \Rightarrow P \leq 1 \Rightarrow P-1 \leq 0$$

$$\Rightarrow P-4 \leq -3 < 0$$

$$P-1 \leq 0; P-4 < 0 \Rightarrow (P-1)(P-4) \geq 0$$

$$P^2 - 5P + 4 \geq 0 \Rightarrow 2P^2 - 10P + 8 \geq 0$$

$$16 + 4P^2 - 16P - 2P^2 + 6P \geq 8$$

$$(4-2P)^2 - 2P^2 + 6P \geq 8$$

$$(S^2 - 2P)^2 - 2P^2 + 6P \geq 8$$

$$a^4 + b^4 + 6ab \geq 8$$

Equality holds for $a = b = 1$.

22. In ΔABC the following relationship holds:

$$\frac{h_a h_b}{r_c^2} + \frac{h_b h_c}{r_a^2} + \frac{h_c h_a}{r_b^2} \geq 12\frac{r^2}{R^2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{h_a h_b}{r_c^2} + \frac{h_b h_c}{r_a^2} + \frac{h_c h_a}{r_b^2} \geq 3\sqrt[3]{\left(\frac{h_a h_b h_c}{r_a r_b r_c}\right)^2} =$$

$$= 3\sqrt[3]{\left(\frac{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}\right)^2} = 3\sqrt[3]{\left(\frac{8(s-a)(s-b)(s-c)}{abc}\right)^2} =$$

$$= 3\sqrt[3]{\left(\frac{8s(s-a)(s-b)(s-c)}{sabc}\right)^2} = 3\sqrt[3]{\left(\frac{8F^2}{s\cdot 4RF}\right)^2} =$$

$$= 3\sqrt[3]{\left(\frac{2F}{Rs}\right)^2} = 3\sqrt[3]{\left(\frac{2rs}{Rs}\right)^2} = 3\sqrt[3]{4\left(\frac{r}{R}\right)^2} \geq 12\left(\frac{r}{R}\right)^2 \Leftrightarrow 27\cdot 4\left(\frac{r}{R}\right)^2 \geq 12^3\cdot\left(\frac{r}{R}\right)^6$$

$$3^3\cdot 4 \geq 3^3\cdot 4^3\cdot\left(\frac{r}{R}\right)^4$$

$$\frac{1}{16} \geq \left(\frac{r}{R}\right)^4 \Leftrightarrow \frac{1}{2} \geq \frac{r}{R} \Leftrightarrow R \geq 2r \qquad \text{(Euler)}$$

Equality holds for $a = b = c$.

23. In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\cot A}{\cot B} \geq \sum_{cyc}\frac{\sin^2 A}{\sin^2 B}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\text{WLOG: } a \le b \le c \Rightarrow \begin{cases} b-a \ge b-c \ge a-c \\ \cot A \le \cot B \le \cot C \end{cases}$$

$$\sum_{cyc}(b-a)\cot A \overset{\text{CEBYSHEV}}{\ge} \frac{1}{3}\sum_{cyc}(b-a)\cdot\sum_{cyc}\cot A = 0$$

$$\sum_{cyc}(b-a)\cot A \ge 0$$

$$\sum_{cyc}(b+a)(b-a)\cot A \ge 0 \Rightarrow \sum_{cyc}(b^2-a^2)\cot A \ge 0$$

$$\Rightarrow \sum_{cyc}(b^2-a^2)\cdot\frac{b^2+c^2-a^2}{4F} \ge 0, \qquad \sum_{cyc}(b^2-a^2)(b^2+c^2-a^2) \ge 0$$

$$\sum_{cyc}(b^4+b^2c^2-b^2a^2-a^2b^2-a^2c^2+a^4) \ge 0$$

$$\sum_{cyc}(a^2(b^2+c^2-a^2)-b^2(a^2+c^2-b^2)) \ge 0$$

$$\sum_{cyc}\frac{b^2+c^2-a^2}{a^2+c^2-b^2} \ge \sum_{cyc}\frac{b^2}{a^2}$$

$$\sum_{cyc}\frac{\cot A}{\cot B} \ge \sum_{cyc}\frac{\sin^2 A}{\sin^2 B}$$

Equality holds for $a = b = c$.

24. In ΔABC the following relationship holds:

$$\frac{\sin A+\sin B}{\cos\frac{C}{2}} + \frac{\sin B+\sin C}{\cos\frac{A}{2}} + \frac{\sin C+\sin A}{\cos\frac{B}{2}} \le 6$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sum_{cyc}\frac{\sin A+\sin B}{\cos\frac{C}{2}} = \sum_{cyc}\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{\cos\frac{C}{2}} =$$

$$= 2\sum_{cyc}\frac{\sin\frac{\pi-C}{2}\cos\frac{A-B}{2}}{\cos\frac{C}{2}} = 2\sum_{cyc}\frac{\cos\frac{C}{2}\cos\frac{A-B}{2}}{\cos\frac{C}{2}} =$$

$$= 2\sum_{cyc}\cos\frac{A-B}{2} \overset{\text{JENSEN}}{\le} 2\cdot 3\cdot\cos\left(\frac{\frac{A-B}{2}+\frac{B-C}{2}+\frac{C-A}{2}}{3}\right) =$$

$$= 6\cos\left(\frac{0}{3}\right) = 6\cos 0 = 6\cdot 1 = 6$$

Equality holds for $A = B = C$.

25. In ΔABC the following relatinship holds:

$$\frac{a^4+b^4}{h_c^3}+\frac{b^4+c^4}{h_a^3}+\frac{c^4+a^4}{h_b^3}\geq 32r$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a^4+b^4}{h_c^3}+\frac{b^4+c^4}{h_a^3}+\frac{c^4+a^4}{h_b^3}=\sum_{cyc}\frac{a^4+b^4}{h_c^3}\geq$$

$$\overset{\text{AM-GM}}{\geq} 2\sum_{cyc}\frac{a^2b^2}{h_c^3}\overset{\text{AM-GM}}{\geq} 2\cdot 3\sqrt[3]{\frac{(abc)^4}{(h_ah_bh_c)^3}}=\frac{6abc}{h_ah_bh_c}\sqrt[3]{abc}=$$

$$=\frac{6abc}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}\sqrt[3]{4RF}=\frac{6(abc)^2}{8F^3}\sqrt[3]{4Rrs}\geq$$

$$\overset{\text{EULER}}{\geq}\frac{6\cdot 16R^2F^2}{8F^3}\sqrt[3]{8r^2s}\overset{\text{MITRINOVIC}}{\geq}\frac{12R^2}{F}\sqrt[3]{8r^2\cdot 3\sqrt{3}r}=$$

$$=\frac{12R^2}{rs}\cdot 2\sqrt{3}r=\frac{24\sqrt{3}R^2}{s}\overset{\text{MITRINOVIC}}{\geq}\frac{24\sqrt{3}\cdot R^2}{\frac{3\sqrt{3}}{2}R}=16R\overset{\text{EULER}}{\geq}16\cdot 2r=32r$$

Equality holds for $a=b=c$.

26. In ΔABC the following relationship holds:

$$\sqrt{a}\sin A+\sqrt{b}\sin B+\sqrt{c}\sin C\leq\frac{3}{2}\sqrt[4]{27R^2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sqrt{a}\sin A+\sqrt{b}\sin B+\sqrt{c}\sin C\overset{\text{CBS}}{\leq}$$

$$\leq\sqrt{(a+b+c)(\sin^2 A+\sin^2 B+\sin^2 C)}=\sqrt{2s\cdot\frac{s^2-r^2-4Rr}{2R^2}}\overset{\text{GERRETSEN}}{\leq}$$

$$=\sqrt{\frac{s}{R^2}(4R^2+4Rr+3r^2-r^2-4Rr)}\overset{\text{MITRINOVIC}}{\leq}\sqrt{\frac{3\sqrt{3}}{2}R\cdot\frac{1}{R^2}\cdot(4R^2+2r^2)}\overset{\text{EULER}}{\leq}$$

$$\leq\sqrt{\frac{3\sqrt{3}}{2R}\left(4R^2+2\cdot\frac{R^2}{4}\right)}=\sqrt{\frac{3\sqrt{3}}{2R}\cdot\frac{9R^2}{2}}=\frac{3}{2}\sqrt{3\sqrt{3}R}=\frac{3}{2}\sqrt[4]{27R^2}$$

Equality holds for $a=b=c$.

27. Solve for real numbers:

$$(x+1)(x+2)^2(x+3)=56$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$(x+1)(x+3)(x+2)^2 = 56$$

$$(x^2+4x+3)(x^2+4x+4) - 56 = 0$$

Denote: $y = x^2 + 4x$

$$(y+3)(y+4) - 56 = 0$$

$$y^2 + 7y - 44 = 0$$

$$\Delta = 49 - 179 = 225$$

$$y_1 = \frac{-7-15}{2} = -11, y_2 = \frac{-7+15}{2} = 4$$

$$x^2 + 4x = -11 \Rightarrow x^2 + 4x + 11 = 0; \Delta < 0$$

$$x^2 + 4x = 4 \Rightarrow x^2 + 4x - 4 = 0$$

$$x_{1,2} = \frac{-4 \pm \sqrt{16+16}}{2} = \frac{-4 \pm 4\sqrt{2}}{2} = -2 \pm 2\sqrt{2}$$

28. In ΔABC the following relationship holds:

$$\sqrt{a \sin^3 A} + \sqrt{b \sin^3 B} + \sqrt{c \sin^3 C} \leq \frac{9\sqrt{2}R}{4}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sqrt{a \sin^3 A} + \sqrt{b \sin^3 B} + \sqrt{c \sin^3 C} = \sum_{cyc} \sqrt{a \sin^3 A} =$$

$$= \sum_{cyc} \sqrt{2R \sin^4 A} = \sqrt{2R} \sum_{cyc} \sin^2 A = \sqrt{2R} \sum_{cyc} \frac{a^2}{4R^2} =$$

$$= \frac{\sqrt{2R}}{4R^2} \cdot \sum_{cyc} a^2 = \frac{\sqrt{2R}}{4R^2} \cdot 2(s^2 - r^2 - 4Rr) \leq$$

$$\overset{\text{GERRETSEN}}{\leq} \frac{\sqrt{2R}}{2R^2}(4R^2 + 4Rr + 3r^2 - r^2 - 4Rr) =$$

$$= \frac{\sqrt{2R}}{2R^2}(4R^2 + 2r^2) \overset{\text{EULER}}{\leq} \frac{\sqrt{2R}}{2R^2}\left(4R^2 + 2 \cdot \frac{R^2}{4}\right) = \frac{\sqrt{2R}}{2R^2} \cdot \frac{18R^2}{4} = \frac{9\sqrt{2R}}{4}$$

Equality holds for $a = b = c$.

29. In ΔABC the following relationship holds:

$$\sqrt{a \cos \frac{A}{2}} + \sqrt{b \cos \frac{B}{2}} + \sqrt{c \cos \frac{C}{2}} \leq \frac{3\sqrt{6}R}{2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sqrt{a\cos\frac{A}{2}}+\sqrt{b\cos\frac{B}{2}}+\sqrt{c\cos\frac{C}{2}}\overset{\text{CBS}}{\leq}$$

$$\leq\sqrt{\sum_{cyc}a\cdot\sum_{cyc}\cos\frac{A}{2}}\overset{\text{JENSEN}}{\leq}\sqrt{2s\cdot 3\cos\left(\frac{A+B+C}{6}\right)}=$$

$$=\sqrt{6s\cdot\cos\frac{\pi}{6}}=\sqrt{6s\cdot\frac{\sqrt{3}}{2}}=\sqrt{3\sqrt{3}s}$$

Remains to prove:

$$\sqrt{3\sqrt{3}s}\leq\frac{3\sqrt{6R}}{2}\Leftrightarrow 3\sqrt{3}s\leq\frac{54R}{2}\Leftrightarrow 3\sqrt{3}s\leq 27R\Leftrightarrow s\leq\frac{9R}{\sqrt{3}}\Leftrightarrow s\leq\frac{3\sqrt{3}}{2}R\ \text{(MITRINOVIC)}$$

Equality holds for $a=b=c$.

30. In ΔABC the following relationship holds:

$$a\sin\frac{A}{2}+\sqrt{b\sin\frac{B}{2}}+\sqrt{c\sin\frac{C}{2}}\leq\frac{3\sqrt[4]{12R^2}}{2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sqrt{a\sin\frac{A}{2}}+\sqrt{b\sin\frac{B}{2}}+\sqrt{c\sin\frac{C}{2}}\overset{\text{CBS}}{\leq}$$

$$\leq\sqrt{\sum_{cyc}a\cdot\sum_{cyc}\sin\frac{A}{2}}\overset{\text{JENSEN}}{\geq}\sqrt{2s\cdot 3\sin\left(\frac{A+B+C}{6}\right)}=$$

$$=\sqrt{6s\cdot\sin\frac{\pi}{6}}=\sqrt{6s\cdot\frac{1}{2}}=\sqrt{3}s$$

Remains to prove:

$$\sqrt{3s}\leq\frac{3\cdot\sqrt[4]{12R^2}}{2}\Leftrightarrow 3s\leq\frac{9\sqrt{12R^2}}{4}\Leftrightarrow 4s\leq 3\sqrt{12R^2}\Leftrightarrow s\leq\frac{3\cdot 2\sqrt{3}R}{4}\Leftrightarrow s\leq\frac{3\sqrt{3}R}{2}\ \text{(MITRINOVIC)}$$

Equality holds for $a=b=c$.

31. In ΔABC the following relationship holds:

$$\sqrt{a\sin A}+\sqrt{b\sin B}+\sqrt{c\sin C}\leq\frac{3\sqrt{6}R}{2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sqrt{a\sin A}+\sqrt{b\sin B}+\sqrt{c\sin C}=\sum_{cyc}\sqrt{a\sin A}=$$

$$=\sum_{cyc}\sqrt{a\cdot\frac{a}{2R}}=\sum_{cyc}\frac{a}{\sqrt{2R}}=\frac{2s}{\sqrt{2R}}=\frac{s\sqrt{2}}{\sqrt{R}}$$

Remains to prove:

$$\frac{s\sqrt{2}}{\sqrt{R}}\leq\frac{3\sqrt{6R}}{2}\Leftrightarrow\frac{2s^2}{R}\leq\frac{9\cdot 6R}{4}$$

$$8s^2 \leq 54R^2 \Leftrightarrow s^2 \leq \frac{27R^2}{4} \Leftrightarrow s \leq \frac{3\sqrt{3}R}{2} \text{ (MITRINOVIC)}$$

Equality holds for $a = b = c$.

32. Solve for real numbers:

$$\sqrt[3]{(1+x)^2} + \sqrt[3]{(1-x)^2} - \sqrt[3]{1-x^2} = 1$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\text{Denote: } a = \sqrt[3]{1+x}; b = \sqrt[3]{1-x}$$

$$a^3 + b^3 = 1 + x + 1 - x = 2 \Rightarrow a^3 + b^3 = 2$$

$$a^2 + b^2 - ab = 1$$

$$(a+b)(a^2 + b^2 - ab) = 2 \Rightarrow a + b = 2$$

$$a^2 + b^2 - ab = 1 \Rightarrow S^2 - 2P - P = 1$$

$$\text{where: } S = a + b; P = ab.$$

$$4 - 3P = 1 \Rightarrow 3P = 3 \Rightarrow P = 1; S = 2$$

$$\left.\begin{aligned} S = a + b = 2 \\ P = ab = 1 \end{aligned}\right\} \Rightarrow a = b = 1$$

$$\sqrt[3]{1+x} = 1 \Rightarrow 1 + x = 1 \Rightarrow x = 0$$

$$\sqrt[3]{1-x} = 1 \Rightarrow 1 - x = 1 \Rightarrow x = 0$$

33. If $-1 \leq x \leq 1$ then:

$$-\frac{4\sqrt{10}}{5} \leq 3x + \sqrt{1-x^2} \leq \sqrt{10}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\text{Let be } f : [-1,1] \to \mathbb{R}; f(x) = 3x + \sqrt{1-x^2}$$

$$f'(x) = 3 - \frac{2x}{2\sqrt{1-x^2}} = 3 - \frac{x}{\sqrt{1-x^2}}$$

$$f'(x) = 0 \Rightarrow 3 - \frac{x}{\sqrt{1-x^2}} = 0 \Leftrightarrow 3 = \frac{x}{\sqrt{1-x^2}}$$

$$\Leftrightarrow 9 = \frac{x^2}{1-x^2} \Leftrightarrow 9 - 9x^2 = x^2 \Leftrightarrow 10x^2 = 9$$

$$\Leftrightarrow x^2 = \frac{9}{10} \Leftrightarrow = \frac{\pm 3\sqrt{10}}{10}$$

$$M = \max_{x \in [-1,1]} f(x) = f\left(\frac{3\sqrt{10}}{10}\right) = 3 \cdot \frac{3\sqrt{10}}{10} + \sqrt{1 - \frac{9}{10}} =$$

$$= \frac{9\sqrt{10}}{10} + \frac{1}{\sqrt{10}} = \frac{9\sqrt{10}}{10} + \frac{\sqrt{10}}{10} = \sqrt{10}$$

$$m = \min_{x \in [-1,1]} f(x) = f\left(-\frac{3\sqrt{10}}{10}\right) = 3 \cdot \frac{-3\sqrt{10}}{10} + \sqrt{1 - \frac{9}{10}} =$$

$$= \frac{-9\sqrt{10}}{10} + \frac{\sqrt{10}}{10} = \frac{-8\sqrt{10}}{10} = \frac{-4\sqrt{10}}{5}$$

$$\Rightarrow -\frac{-4\sqrt{10}}{5} \le f(x) \le \sqrt{10}$$

$$\frac{-4\sqrt{10}}{5} \le 3x + \sqrt{1-x^2} \le \sqrt{10}$$

34. If $a, b > 0; a + b = 2$ then:

$$5(a^2+b^2) + \frac{4(ab+a+b)}{a^2+b^2+1} \ge 14$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\text{Denote: } S = a+b; P = ab \Rightarrow S = 2$$

$$2 = S = a + b \overset{\text{AM-GM}}{\ge} 2\sqrt{ab} = 2\sqrt{P} \Rightarrow \sqrt{P} \le 1$$

$$\Rightarrow P \le 1 \Rightarrow P - 1 \le 0 \tag{1}$$

$$P \le 1 \Rightarrow 10P \le 10 \Rightarrow 10P - 19 \le 10 - 19 = -9 < 0$$

$$\Rightarrow 10P - 19 < 0 \tag{2}$$

$$\text{By (1); (2) } \Rightarrow (P-1)(10P-19) \ge 0 \tag{3}$$

$$5(a^2+b^2) + \frac{4(ab+a+b)}{a^2+b^2+1} \ge 14 \Rightarrow$$

$$\Rightarrow 5(S^2-2P) + \frac{4(P+S)}{S^2-2P+1} - 14 \ge 0$$

$$\Rightarrow 5(4-2P) + \frac{4(P+2)}{5-2P} - 14 \ge 0$$

$$20 - 10P + \frac{4P+8}{5-2P} - 14 \ge 0$$

$$6 - 10P \ge \frac{4P+8}{2P-5} \Leftrightarrow 3 - 5P \ge \frac{2P+4}{2P-5} \Leftrightarrow$$

$$(3-5P)(2P-5) \le 2P+4 \Leftrightarrow 6P - 15 - 10P^2 + 25P \le 2P + 4$$

$$\Leftrightarrow -10P^2 + 31P - 15 - 2P - 4 \le 0$$

$$-10P^2 + 29P - 19 \le 0$$

$$10P^2 - 29P + 19 \ge 0$$

$$(P-1)(10P-19) \ge 0 \text{ which is true: (3)}$$

Equality holds for $a = b = 1$.

35. Solve for real numbers:

$$\sqrt[4]{x - \sqrt{x^2-1}} + \sqrt{x + \sqrt{x^2-1}} = 2$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania
Denote:

$$a = \sqrt[4]{x - \sqrt{x^2 - 1}}; b = \sqrt[4]{x + \sqrt{x^2 - 1}}$$

$$a \cdot b = \sqrt[4]{x^2 - (\sqrt{x^2 - 1})^2} = \sqrt[4]{x^2 - x^2 + 1} = 1$$

$$b = \frac{1}{a}; a + b^2 = 2 \Rightarrow a + \frac{1}{a^2} = 2$$

$$a^3 + 1 - 2a^2 = 0 \Rightarrow a^3 - 2a^2 + 1 = 0$$

$$a^3 - a^2 - (a^2 - 1) = 0 \Rightarrow a^2(a - 1) - (a - 1)(a + 1) = 0$$

$$(a - 1)(a^2 - a - 1) = 0$$

$$a - 1 = 0 \Rightarrow a = 1 \Rightarrow \sqrt[4]{x - \sqrt{x^2 - 1}} = 1$$

$$x - \sqrt{x^2 - 1} = 1 \Rightarrow x - 1 = \sqrt{x^2 - 1} \Rightarrow x^2 - 2x + 1 = x^2 - 1$$

$$-2x = -2 \Rightarrow x = 1$$

$$a^2 - a - 1 = 0 \Rightarrow a_{2,3} = \frac{1 \pm \sqrt{5}}{2}$$

$$\sqrt[4]{x - \sqrt{x^2 - 1}} = a \Rightarrow x - \sqrt{x^2 - 1} = a^2$$

$$x - a^2 = \sqrt{x^2 - 1} \Rightarrow x^2 - 2xa^2 + a^4 = x^2 - 1$$

$$2xa^2 = a^4 + 1 \Rightarrow x = \frac{a^4 + 1}{2a^2} \Rightarrow x = \frac{a^2}{2} + \frac{1}{2a^2}$$

$$x_1 = 1; x_{2,3} = \frac{a^2}{2} + \frac{1}{2a^2}; a \in \left\{ \frac{1 \pm \sqrt{5}}{2} \right\}$$

36. Solve for real numbers:

$$(x^2 - 3x + 1)^2 = 2x - 1$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$(x^2 - 3x + 1)^2 = 2x - 1$$

$$(x^2 - 3x + 1)^2 - x^2 = -x^2 + 2x - 1$$

$$(x^2 - 3x + 1 - x)(x^2 - 3x + 1 + x) + (x^2 - 2x + 1) = 0$$

$$(x^2 - 4x + 1)(x - 1)^2 + (x - 1)^2 = 0$$

$$(x - 1)^2(x^2 - 4x + 2) = 0$$

$$x - 1 = 0 \Rightarrow x_1 = x_2 = 1$$

$$x^2 - 4x + 2 = 0 \Rightarrow \begin{cases} x_3 = \frac{4+\sqrt{8}}{2} = 2 + \sqrt{2} \\ x_4 = \frac{4-\sqrt{8}}{2} = 2 - \sqrt{2} \end{cases}$$

37. In ΔABC the following relationship holds:

$$\cos A \cos B \cos C \leq \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania
We use:

$$\cos A \cos B \cos C = \frac{s^2 - (2R + r)^2}{4R^2}$$
$$\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{r}{4R}$$

We must prove:

$$\frac{s^2 - (2R + r)^2}{4R^2} \leq \frac{r}{4R}$$
$$s^2 - (2R + r)^2 \leq Rr$$
$$s^2 \leq 4R^2 + 4Rr + r^2 + Rr = 4R^2 + 5Rr + r^2$$
$$s^2 \overset{\text{GERRETSEN}}{\leq} 4R^2 + 4Rr + 3r^2 \leq 4R^2 + 5Rr + r^2 \Leftrightarrow$$
$$\Leftrightarrow 3r^2 \leq Rr + r^2 \Leftrightarrow Rr \geq 2r^2$$
$$\Leftrightarrow R \geq 2R \text{ (Euler)}$$

Equality holds for $a = b = c$.

38. In ΔABC the following relationship holds:

$$\sin A \sin B \sin C \leq \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania
We use:

$$\sin A \sin B \sin C = \frac{sr}{2R^2}; \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \frac{s}{4R}$$

We must prove that:

$$\frac{sr}{2R^2} \leq \frac{s}{4R} \Leftrightarrow \frac{r}{2R^2} \leq \frac{1}{4R} \Leftrightarrow 2r \leq R \text{ (Euler)}$$

Equality holds for $A = B = C$.

39. In ΔABC the following relationship holds:

$$\frac{\sin A}{r_a} + \frac{\sin B}{r_b} + \frac{\sin C}{r_c} \leq \frac{\sqrt{3}}{2r}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{\sin A}{r_a} + \frac{\sin B}{r_b} + \frac{\sin C}{r_c} = \sum_{cyc} \frac{\sin A}{r_a} = \sum_{cyc} \frac{\frac{a}{2R}}{\frac{F}{s-a}} =$$
$$= \frac{1}{2RF} \sum_{cyc} a(s-a) = \frac{1}{2RF} \left(s \sum_{cyc} a - \sum_{cyc} a^2 \right) =$$
$$= \frac{1}{2RF} (s \cdot 2s - 2s^2 + 2r^2 + 8Rr) = \frac{2r(r + 4R)}{2RF} =$$
$$= \frac{r(r + 4R)}{Rrs} = \frac{r + 4R}{Rs} \overset{\text{EULER}}{\leq} \frac{\frac{R}{2} + 4R}{Rs} =$$
$$= \frac{\frac{1}{2} + 4}{s} \overset{\text{MITRINOVIC}}{\leq} \frac{\frac{9}{2}}{3\sqrt{3}r} = \frac{3}{2\sqrt{3}r} = \frac{\sqrt{3}}{2r}$$

Equality holds for $a = b = c$.

40. In ΔABC the following relationship holds:

$$\frac{\sin A}{r_b} + \frac{\sin B}{r_c} + \frac{\sin C}{r_a} \geq \frac{\sqrt{3}}{R}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{\sin A}{r_b} + \frac{\sin B}{r_c} + \frac{\sin C}{r_a} \geq 3\sqrt[3]{\frac{\sin A \sin B \sin C}{r_a r_b r_c}} =$$
$$= 3\sqrt[3]{\frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} \cdot \frac{(s-a)(s-b)(s-c)}{F^3}} =$$
$$= \frac{3}{2R}\sqrt[3]{abc \cdot \frac{F^2}{sF^3}} = \frac{3}{2R}\sqrt[3]{\frac{abc}{sF}} =$$
$$= \frac{3}{2R} \cdot \sqrt[3]{\frac{4RF}{sF}} = \frac{3}{2R}\sqrt[3]{\frac{4R}{s}} \geq \frac{3}{2R}\sqrt[3]{4 \cdot \frac{2}{3\sqrt{3}} \cdot \frac{s}{s}} =$$
$$= \frac{3}{2R} \cdot \frac{2}{\sqrt{3}} = \frac{\sqrt{3}}{R}$$

Equality holds for $a = b = c$.

41. Solve for real numbers:

$$\sqrt[3]{1+x} + \sqrt[3]{1-x} = 2\sqrt[3]{1-x^2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\text{Denote: } \begin{cases} \sqrt[3]{1+x} = a \\ \sqrt[3]{1-x} = b \end{cases} \Rightarrow a^3 + b^3 = 2$$

$$\begin{cases} a^3 + b^3 = 2 \\ a + b = 2ab \end{cases}. \text{ Denote: } \begin{cases} S = a + b \\ P = ab \end{cases} \Rightarrow \begin{cases} S^3 - 3SP = 2 \\ S = 2P \end{cases}$$

$$\Rightarrow (2P)^3 - 3 \cdot 2P \cdot P = 2$$
$$8P^3 - 6P^2 - 2 = 0 \Rightarrow 4P^3 - 3P^2 - 1 = 0$$
$$4P^3 - 4P^2 + P^2 - 1 = 0 \Rightarrow 4P^2(P-1) + (P-1)(P+1) = 0$$
$$(P-1)(4P^2 + P + 1) = 0$$
$$a, b > 0 \Rightarrow P = ab > 0 \Rightarrow 4P^2 + P + 1 > 0$$
$$\Rightarrow P - 1 = 0 \Rightarrow P = 1 \Rightarrow S = 2 \cdot 1 \Rightarrow S = 2$$
$$\begin{cases} S = 2 \\ P = 1 \end{cases} \Rightarrow a = b = 1 \Rightarrow 1 + x = 1 - x = 1 \Rightarrow x = 0$$

42. Solve for real numbers:

$$\sqrt{1+x} + \sqrt{1-x} + 2x^2 = 2$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\text{Denote: } \begin{cases} \sqrt{1+x} = a \\ \sqrt{1-x} = b \end{cases} \Rightarrow \begin{cases} 1+x = a^2 \\ 1-x = b^2 \end{cases} \Rightarrow a^2+b^2 = 2$$

$$2-2x^2 = 2(1-x^2) = 2(1-x)(1+x) = 2a^2b^2$$

$$\begin{cases} a^2+b^2 = 2 \\ a+b = 2a^2b^2 \end{cases}. \text{ Denote: } \begin{cases} S = a+b \\ P = ab \end{cases}$$

$$\Rightarrow \begin{cases} S^2-2P = 2 \\ S = 2P^2 \end{cases} \Rightarrow (2P^2)^2 - 2P = 0$$

$$4P^4 - 4P + 2P - 2 = 0 \Rightarrow 4P(P^3-1) + 2(P-1) = 0$$

$$4P(P-1)(P^2+P+1) + 2(P-1) = 0$$

$$2(P-1)(2P^3+2P^2+2P+1) = 0$$

$$a, b > 0 \Rightarrow ab > 0 \Rightarrow P > 0 \Rightarrow 2P^3+2P^2+2P+1 > 0$$

$$\Rightarrow P-1 = 0 \Rightarrow P = 1 \Rightarrow S = 2\cdot 1^2 = 2$$

$$\begin{cases} S = 2 \\ P = 1 \end{cases} \Rightarrow a = b = 1 \Rightarrow 1-x = 1+x = 1$$

$$\Rightarrow x = 0$$

43. In ΔABC the following relationship holds:

$$\frac{ab}{\sin\frac{C}{2}} + \frac{bc}{\sin\frac{A}{2}} + \frac{ca}{\sin\frac{B}{2}} \geq 72r^2$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{ab}{\sin\frac{C}{2}} + \frac{bc}{\sin\frac{A}{2}} + \frac{ca}{\sin\frac{B}{2}} = \sum_{cyc} \frac{ab}{\sin\frac{C}{2}} \geq$$

$$\overset{\text{CEBYSEV}}{\geq} \frac{1}{3}\sum_{cyc} ab \cdot \sum_{cyc} \frac{1}{\sin\frac{C}{2}} \overset{\text{JENSEN}}{\geq}$$

$$\geq \frac{1}{3}(s^2+r^2+4Rr)\cdot \frac{3}{\sin\left(\frac{A+B+C}{6}\right)} \overset{\text{MITRINOVIC}}{\geq}$$

$$\geq (27r^2+r^2+4R+1)\cdot \frac{1}{\sin\frac{\pi}{6}} \overset{\text{EULER}}{\geq}$$

$$\geq (28r^2 + 4\cdot 2r\cdot r)\cdot \frac{1}{\frac{1}{2}} = 2(28r^2+8r^2) =$$

$$= 2\cdot 36r^2 = 72r^2$$

Equality holds for $a = b = c$.

44. In ΔABC the following relationship holds:

$$\frac{a}{h_a \sin A} + \frac{b}{h_b \sin B} + \frac{c}{h_c \sin C} \geq 4$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a}{h_a \sin A}+\frac{b}{h_b \sin B}+\frac{c}{h_c \sin C}=\sum_{cyc}\frac{a}{h_a \sin A}=$$

$$=\sum_{cyc}\frac{a}{\frac{2F}{a}\cdot\frac{a}{2R}}=\frac{R}{F}\sum_{cyc}a=\frac{R}{F}\cdot 2s=$$

$$=\frac{R}{rs}\cdot 2s=\frac{2R}{r}\overset{\text{EULER}}{\geq}\frac{2\cdot 2r}{r}=4$$

Equality holds for $a=b=c$.

45. In ΔABC the following relationship holds:

$$\frac{a}{h_a \sin\frac{A}{2}}+\frac{b}{h_b \sin\frac{B}{2}}+\frac{c}{h_c \sin\frac{C}{2}}\geq 4\sqrt{3}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a}{h_a \sin\frac{A}{2}}+\frac{b}{h_b \sin\frac{B}{2}}+\frac{c}{h_c \sin\frac{C}{2}}=\sum_{cyc}\frac{a}{h_a \sin\frac{A}{2}}=$$

$$=\sum_{cyc}\frac{a}{\frac{2F}{a}\sin\frac{A}{2}}=\frac{1}{2F}\sum_{cyc}\frac{a^2}{\sin\frac{A}{2}}\overset{\text{CEBYSHEV}}{\geq}$$

$$\geq\frac{1}{2F}\cdot\frac{1}{3}\sum_{cyc}a^2\sum_{cyc}\frac{1}{\sin\frac{A}{2}}\overset{\text{JENSEN}}{\geq}$$

$$\geq\frac{1}{6F}\cdot\sum_{cyc}a^2\cdot\frac{3}{\sin\left(\frac{A+B+C}{6}\right)}=\frac{1}{2F}\sum_{cyc}a^2\cdot\frac{1}{\frac{1}{2}}=$$

$$=\frac{1}{rs}\cdot 2(s^2-r^2-4Rr)\overset{\text{GERRETSEN}}{\geq}$$

$$\geq\frac{2}{rs}(16Rr-5r^2-r^2-4Rr)=\frac{2(12Rr-6r^2)}{rs}=$$

$$=\frac{2(12R-6r)}{s}=\frac{12(2R-r)}{s}\geq 4\sqrt{3}\Leftrightarrow$$

$$\Leftrightarrow 3(2R-r)\geq s\sqrt{3}\text{ (to prove)}$$

$$s\sqrt{3}\overset{\text{DOUCET}}{\leq}4R+r\leq 3(2R-r)$$

$$4R+r\leq 6R-3r$$

$$R\geq 2r\text{ (Euler)}$$

Equality holds for $a=b=c$.

46. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin A}{h_b}\geq\frac{\sqrt{3}}{R}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sum_{cyc} \frac{\sin A}{h_b} = \sum_{cyc} \frac{\frac{a}{2R}}{\frac{2F}{b}} = \frac{1}{4RF} \sum_{cyc} ab =$$

$$= \frac{1}{4Rrs} \cdot (s^2 + r^2 + 4Rr) \overset{\text{MITRINOVIC}}{\geq} \frac{1}{4Rr \cdot \frac{3\sqrt{3}}{2} R}(s^2 + r^2 + 4Rr) \geq$$

$$\overset{\text{GERRETSEN}}{\geq} \frac{1}{6\sqrt{3}R^2 r}(16Rr - 5r^2 + r^2 + 4Rr) =$$

$$= \frac{1}{6\sqrt{3}R^2 r}(20Rr - 4r^2) = \frac{2r(10R - 2r)}{6\sqrt{3}R^2 r} =$$

$$= \frac{10R - 2r}{3\sqrt{3}R^2} \geq \frac{\sqrt{3}}{R} \Leftrightarrow 10R - 2r \geq 9R$$

$$\Leftrightarrow R \geq 2r \text{ (Euler)}$$

Equality holds for $a = b = c$.

47. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\sin A}{h_a} \geq \frac{\sqrt{3}}{R}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sum_{cyc} \frac{\sin A}{h_a} = \sum_{cyc} \frac{\frac{a}{2R}}{\frac{2F}{a}} = \frac{1}{4RF} \sum_{cyc} a^2 =$$

$$= \frac{1}{4Rrs} \cdot 2(s^2 - r^2 - 4Rr) \overset{\text{MITRINOVIC}}{\geq}$$

$$\geq \frac{1}{2Rr \cdot \frac{3\sqrt{3}R}{2}}(s^2 - r^2 - 4Rr) = \frac{1}{3\sqrt{3}R^2 r}(s^2 - r^2 - 4Rr) \geq$$

$$\overset{\text{GERRETSEN}}{\geq} \frac{1}{3\sqrt{3}R^2 r}(16Rr - 5r^2 - r^2 - 4Rr) = \frac{6r(2R - r)}{3\sqrt{3}R^2 r} =$$

$$= \frac{2(2R - r)}{\sqrt{3}R^2} \geq \frac{\sqrt{3}}{R} \Leftrightarrow 2(2R - r) \geq 3R$$

$$4R - 2r \geq 3R \Leftrightarrow R \geq 2r \text{ (Euler)}$$

Equality holds for $a = b = c$.

48. Solve for real numbers:

$$(x+1)(x+3)(x+5)(x+7) = 9$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$(x+1)(x+3)(x+5)(x+7)=9$$
$$(x+1)(x+7)(x+3)(x+5)=9$$
$$(x^2+8x+7)(x^2+8x+15)=9$$
$$\text{Denote: } x^2+8x=y \Rightarrow (y+7)(y+15)=9$$
$$y^2+22y+105-9=0 \Rightarrow y^2+22y+96=0$$
$$\Delta=484-384=100$$
$$y_1=\frac{-22-10}{2}=-\frac{32}{2}=-16; y_2=\frac{-22+10}{2}=\frac{-16}{2}=-8$$
$$x^2+8x=16 \Rightarrow x^2+8x+16=0 \Rightarrow (x+4)^2=0$$
$$\Rightarrow x_1=x_2=-4$$
$$x^2+8x=-8 \Rightarrow x^2+8x+8=0$$
$$\Delta=64-32=32$$
$$x_{3,4}=\frac{-8\pm 4\sqrt{2}}{2}=-4\pm 2\sqrt{2}$$

49. Solve for real numbers:

$$\sqrt{x}+\sqrt{1-x}=1+\sqrt{x(1-x)}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$x\ge 0$$
$$1-x\ge 0 \Rightarrow x\in[0,1]$$
$$\text{Denote: } \begin{cases}\sqrt{x}=a\\ \sqrt{1-x}=b\end{cases} \Rightarrow a^2+b^2=1$$
$$\sqrt{x}+\sqrt{1-x}=1+\sqrt{x(1-x)} \Rightarrow a+b=1+ab$$
$$\begin{matrix} ab-a+1-b=0 \\ (b-1)(a-1)=0 \end{matrix} \Rightarrow a(b-1)-(b-1)=0$$
$$a-1=0 \Rightarrow a=1 \Rightarrow 1^2b^2=1 \Rightarrow b=0 \Rightarrow$$
$$\Rightarrow \begin{cases}\sqrt{x}=1\\ \sqrt{1-x}=0\end{cases} \Rightarrow x=1$$
$$b-1=0 \Rightarrow b=1 \Rightarrow a^2+1^2=1 \Rightarrow a=0 \Rightarrow$$
$$\Rightarrow \begin{cases}\sqrt{x}=0\\ \sqrt{1-x}=1\end{cases} \Rightarrow x=0$$
$$\Rightarrow S=\{0,1\}$$

50. In ΔABC the following relationship holds:

$$\frac{ab}{\cos\frac{C}{2}}+\frac{bc}{\cos\frac{A}{2}}+\frac{ca}{\cos\frac{B}{2}}\ge 24\sqrt{3}r^2$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{ab}{\cos\frac{C}{2}}+\frac{bc}{\cos\frac{A}{2}}+\frac{ca}{\cos\frac{B}{2}}=\sum_{cyc}\frac{ab}{\cos\frac{C}{2}}\geq$$

$$\overset{\text{CEBYSHEV}}{\geq}\frac{1}{3}\sum_{cyc}ab\cdot\sum_{cyc}\frac{1}{\cos\frac{C}{2}}\overset{\text{JENSEN}}{\geq}$$

$$\frac{1}{3}\cdot\sum_{cyc}ab\cdot 3\cdot\frac{1}{\cos\left(\frac{A+B+C}{6}\right)}\overset{\text{AM-GM}}{\geq}\frac{1}{\cos\frac{\pi}{6}}\cdot\sum_{cyc}ab\geq$$

$$\geq\frac{1}{\frac{\sqrt{3}}{2}}\cdot 3\cdot\sqrt[3]{(abc)^2}=\frac{6}{\sqrt{3}}\sqrt[3]{(4RF)^2}=$$

$$=\frac{6\sqrt{3}}{3}\sqrt[3]{(4Rrs)^2}\overset{\text{EULER}}{\geq}2\sqrt{3}\cdot\sqrt[3]{(8r^2s)^2}\geq$$

$$\overset{\text{MITRINOVIC}}{\geq}2\sqrt{3}\cdot\sqrt[3]{(8r^2\cdot 3\sqrt{3}r)^2}=$$

$$=2\sqrt{3}\cdot\sqrt[3]{64r^6\cdot 27}=$$

$$=2\sqrt{3}\cdot 4\cdot 3\cdot r^2=24\sqrt{3}r^2$$

Equality holds for $a=b=c$.

51. Solve for real numbers:

$$\sqrt{1+x}-\sqrt{1-x}+\sqrt{1-x^2}=1$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\text{Denote: }\begin{cases}\sqrt{1+x}=a\\ \sqrt{1-x}=b\end{cases}\Rightarrow a^2+b^2=2$$

$$\sqrt{1+x}-\sqrt{1-x}+\sqrt{1-x^2}=1\Rightarrow a-b+ab=1$$

$$a(1+b)-(1+b)=0\Rightarrow(1+b)(a-1)=0$$

If $b+1=0\Rightarrow b>-1$. But $6>0$. Contradiction!

$$a-1=0\Rightarrow a=1\Rightarrow\sqrt{1+x}=1\Rightarrow x=0$$

52. Solve for real numbers:

$$\sqrt[3]{1+x}-\sqrt[3]{1-x}+\sqrt[3]{1-x^2}=1$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\text{Denote: }\begin{cases}a=\sqrt[3]{1+x}\\ b=\sqrt[3]{1-x}\end{cases}\Rightarrow\begin{cases}1+x=a^3\\ 1-x=b^3\end{cases}\Rightarrow a^3+b^3=2$$

$$\sqrt[3]{1+x}-\sqrt[3]{1-x}+\sqrt[3]{1-x^2}=1\Rightarrow a-b+ab=1$$

$$a(1+b)-(1+b)=0\Rightarrow(a-1)(b+1)=0$$

$$a-1=0\Rightarrow a=1\Rightarrow 1^3+b^3=2\Rightarrow b=1$$

$$\Rightarrow \begin{cases} \sqrt[3]{1+x} = 1 \\ \sqrt[3]{1-x} = 1 \end{cases} \Rightarrow \begin{cases} 1+x = 1 \\ 1-x = 1 \end{cases} \Rightarrow x = 0$$

$$b+1 = 0 \Rightarrow b = -1 \Rightarrow a^3 = 3 \Rightarrow a = \sqrt[3]{3}$$

$$\sqrt[3]{1-x} = -1 \Rightarrow 1-x = -1 \Rightarrow x = 2$$

$$\Rightarrow S = \{0, 2\}$$

53. Solve for real numbers:

$$\sqrt[3]{(1+x)^2} + \sqrt[3]{(1-x)^2} - \sqrt[3]{1-x^2} = 1$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\text{Denote: } \begin{cases} \sqrt[3]{1+x} = a \\ \sqrt[3]{1-x} = b \end{cases} \Rightarrow a^3 + b^3 = 1 + x + 1 - x = 2$$

$$\sqrt[3]{(1+x)^2} + \sqrt[3]{(1-x)^2} - \sqrt[3]{1-x^2} = 1 \Rightarrow$$

$$\Rightarrow a^2 + b^2 - ab = 1$$

$$\begin{cases} a^3 + b^3 = 2 \\ a^2 + b^2 - ab = 1 \end{cases} . \text{ Denote: } \begin{cases} S = a + b \\ P = ab \end{cases}$$

$$\Rightarrow \begin{cases} S^3 - 3SP = 2 \\ S^2 - 2P - P = 1 \end{cases} \Rightarrow \begin{cases} S(S^2 - 3P) = 2 \\ S^2 - 3P = 1 \end{cases} \Rightarrow S = 2$$

$$S^2 - 3P = 1 \Rightarrow 4 - 3P = 1 \Rightarrow P = 1$$

$$\begin{cases} S = 2 \\ P = 1 \end{cases} \Rightarrow a = b = 1 \Rightarrow \begin{cases} \sqrt[3]{1+x} = 1 \\ \sqrt[3]{1-x} = 1 \end{cases} \Rightarrow x = 0$$

54. Solve for real numbers:

$$\sqrt[3]{1+x} + \sqrt[3]{1-x} - \sqrt[3]{1-x^2} = 1$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\text{Denote: } \begin{cases} \sqrt[3]{1+x} = a \\ \sqrt[3]{1-x} = b \end{cases} \Rightarrow \begin{cases} a^3 = 1 + x \\ b^3 = 1 - x \end{cases} \Rightarrow$$

$$\Rightarrow a^3 + b^3 = 2$$

$$\sqrt[3]{1+x} + \sqrt[3]{1-x} - \sqrt[3]{1-x^2} = 1 \Rightarrow a + b - ab = 1$$

$$\text{Denote: } S = a + b; P = ab$$

$$\begin{cases} a^3 + b^3 = 2 \\ a + b - ab = 1 \end{cases} \Rightarrow \begin{cases} S^3 - 3SP = 2 \\ S - P = 1 \end{cases}$$

$$P = S - 1 \Rightarrow S^3 - 3S(S-1) = 2$$

$$S^3 - 3S^2 + 3S - 2 = 0$$

$$S^3 - 2S^2 - S^2 + 2S + S - 2 = 0$$

$$S^2(S-2) - S(S-2) + (S-2) = 0$$

$$(S-2)(S^2 - S + 1) = 0$$

$$S-2=0 \Rightarrow S=2 \Rightarrow P=2-1=1$$

$$\begin{cases} S=2 \\ P=1 \end{cases} \Rightarrow a=b=1 \Rightarrow \begin{cases} \sqrt[3]{1+x}=1 \\ \sqrt[3]{1-x}=1 \end{cases} \Rightarrow x=0$$

55. In ΔABC the following relationship holds:

$$\frac{a\sin A}{h_a}+\frac{b\sin B}{h_b}+\frac{c\sin C}{h_c}\geq 3$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a\sin A}{h_a}+\frac{b\sin B}{h_b}+\frac{c\sin C}{h_c}=\sum_{cyc}\frac{a\sin A}{h_a}=$$

$$=\sum_{cyc}\frac{a\cdot\frac{a}{2R}}{\frac{2F}{a}}=\frac{1}{4RF}\sum_{cyc}a^3\overset{\text{AM-GM}}{\geq}$$

$$\geq\frac{1}{abc}\cdot 3\sqrt[3]{(abc)^3}=\frac{3abc}{abc}=3$$

Equality holds for $a=b=c$.

56. In ΔABC the following relationship holds:

$$\frac{ab}{\sin C}+\frac{bc}{\sin A}+\frac{ca}{\sin B}\geq 24\sqrt{3}r^2$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{ab}{\sin C}+\frac{bc}{\sin A}+\frac{ca}{\sin B}=\sum_{cyc}\frac{ab}{\sin C}=\sum_{cyc}\frac{ab}{\frac{c}{2R}}=$$

$$=2R\sum\frac{ab}{c}\overset{\text{AM-GM}}{\geq}2R\cdot 3\sqrt[3]{\frac{ab}{c}\cdot\frac{bc}{a}\cdot\frac{ca}{b}}=$$

$$=6R\sqrt[3]{abc}=6R\sqrt[3]{4Rrs}\overset{\text{EULER}}{\geq}$$

$$\geq 6R\cdot\sqrt[3]{4\cdot 2r\cdot rs}=12R\sqrt[3]{r^2s}\overset{\text{MITRINOVIC}}{\geq}$$

$$\geq 12R\sqrt[3]{r^2\cdot 3\sqrt{3}r}\overset{\text{EULER}}{\geq}12\cdot 2r\cdot\sqrt[3]{(r\sqrt{3})^3}=$$

$$=24r\cdot\sqrt{3}r=24\sqrt{3}r^2$$

Equality holds for $a=b=c$.

57. In ΔABC the following relationship holds:

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\geq 6\sqrt{3}r$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\overset{\text{AM-GM}}{\geq}3\sqrt[3]{\frac{ab}{c}\cdot\frac{bc}{a}\cdot\frac{ca}{b}}=$$

$$=3\cdot\sqrt[3]{abc}=3\sqrt[3]{4Rrs}\overset{\text{EULER}}{\geq}$$

$$=3\sqrt[3]{4\cdot 2r\cdot rs}\overset{\text{MITRINOVIC}}{\geq}3\cdot 2\sqrt[3]{r^2\cdot 3\sqrt{3}r}=$$

$$=6\sqrt[3]{(r\sqrt{3})^3}=6\sqrt{3}r$$

Equality holds for $a=b=c$.

58. In acute ΔABC the following relationship holds:

$$\frac{a}{h_a\cos^2\frac{A}{2}}+\frac{b}{h_b\cos^2\frac{B}{2}}+\frac{c}{h_c\cos^2\frac{C}{2}}\geq\frac{8\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a}{h_a\cos^2\frac{A}{2}}+\frac{b}{h_b\cos^2\frac{B}{2}}+\frac{c}{h_c\cos^2\frac{C}{2}}\overset{\text{AM-GM}}{\geq}$$

$$\geq 3\sqrt[3]{\frac{abc}{h_ah_bh_c\left(\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)^2}}=$$

$$=3\sqrt[3]{\frac{abc}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}\cdot\left(\frac{s}{4R}\right)^2}}=$$

$$=3\sqrt[3]{\frac{(abc)^2\cdot 16R^2}{8F^3\cdot s^2}}=3\sqrt[3]{\frac{16R^2F^2\cdot 2R^2}{F^3s^2}}=$$

$$=3\sqrt[3]{\frac{32R^4}{Fs^2}}\overset{\text{EULER}}{\geq}3\sqrt[3]{\frac{32\cdot 2r\cdot R^3}{rs^3}}=$$

$$=3\sqrt[3]{64\left(\frac{R}{s}\right)^3}=\frac{12R}{s}\overset{\text{MITRINOVIC}}{\geq}$$

$$\geq\frac{12\cdot\frac{2}{3\sqrt{3}}s}{s}=\frac{24}{3\sqrt{3}}=\frac{8}{\sqrt{3}}=\frac{8\sqrt{3}}{3}$$

Equality holds for $a=b=c$.

59. In ΔABC the following relationship holds:

$$\frac{a}{r_a\sin A}+\frac{b}{r_b\sin B}+\frac{c}{r_c\sin C}\geq 4$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a}{r_a \sin A}+\frac{b}{r_b \sin B}+\frac{c}{r_c \sin C}=\sum_{cyc}\frac{a}{r_a \sin A}=$$

$$=\sum_{cyc}\frac{a}{\frac{F}{s-a}\cdot\frac{a}{2R}}=\frac{2R}{F}\sum_{cyc}(s-a)=$$

$$=\frac{2R}{F}\left(\sum_{cyc}s-\sum_{cyc}a\right)=\frac{2R}{rs}(3s-2s)=\frac{2R}{r}\geq$$

$$\overset{\text{EULER}}{\geq}\frac{2\cdot 2r}{r}=4$$

Equality holds for $a=b=c$.

60. In acute ΔABC the following relationship holds:

$$\frac{a}{r_a \cos A}+\frac{b}{r_b \cos B}+\frac{c}{r_c \cos C}\geq 4\sqrt{3}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania
We will use the next known result:

$$\cos A\cos B\cos C\leq\frac{1}{8}\tag{1}$$

$$\frac{a}{r_a \cos A}+\frac{b}{r_b \cos B}+\frac{c}{r_c \cos C}\geq 3\sqrt[3]{\frac{abc}{r_ar_br_c\cos A\cos B\cos C}}\geq$$

$$\overset{(1)}{\geq}3\sqrt[3]{\frac{abc}{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}\cdot\frac{1}{8}}}=3\sqrt[3]{\frac{8abc(s-a)(s-b)(s-c)}{F^3}}=$$

$$=3\sqrt[3]{\frac{8abc\cdot F^2}{sF^3}}=6\sqrt[3]{\frac{4RF}{sF}}=6\sqrt[3]{\frac{4R}{s}}\geq$$

$$\overset{\text{MITRINOVIC}}{\geq}6\sqrt[3]{\frac{4\cdot\frac{2}{3\sqrt{3}}s}{s}}=6\sqrt[3]{\left(\frac{2}{\sqrt{3}}\right)^3}=$$

$$=6\cdot\frac{2}{\sqrt{3}}=\frac{12\sqrt{3}}{3}=4\sqrt{3}$$

Equality holds for $a=b=c$.

61. In any acute ΔABC the following relationship holds:

$$\frac{a\cot A}{h_a}+\frac{b\cot B}{h_b}+\frac{c\cot C}{h_c}=2$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a\cot A}{h_a}+\frac{b\cot B}{h_b}+\frac{c\cot C}{h_c}=\sum_{cyc}\frac{a\cot A}{h_a}=$$

$$=\sum_{cyc}\frac{a}{\frac{2F}{a}}\cdot\frac{\cos A}{\sin A}=\frac{1}{2F}\sum_{cyc}a^2\cdot\frac{b^2+c^2-a^2}{2bc\sin A}=$$

$$=\frac{1}{2F}\cdot\frac{1}{4F}\sum_{cyc}a^2(b^2+c^2-a^2)=$$

$$=\frac{1}{8F^2}(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4)=$$

$$=\frac{1}{8F^2}\cdot 16F^2=2$$

62. In ΔABC the following relationship holds:

$$\frac{a}{h_b^2}+\frac{b}{h_c^2}+\frac{c}{h_a^2}\geq\frac{4\sqrt{3}}{3R}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a}{h_b^2}+\frac{b}{h_c^2}+\frac{c}{h_a^2}\overset{\text{AM-GM}}{\geq}\sqrt[3]{\frac{abc}{(h_ah_bh_c)^2}}=$$

$$=3\sqrt[3]{\frac{abc}{\left(\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}\right)^2}}=3\sqrt[3]{\frac{(abc)^3}{64F^6}}=$$

$$=3\cdot\frac{abc}{4F^2}=3\cdot\frac{4RF}{4F^2}=\frac{3R}{F}=\frac{3R}{rs}\geq$$

$$\overset{\text{MITRINOVIC}}{\geq}\frac{3R}{r\cdot\frac{3\sqrt{3}}{2}R}=\frac{2}{r\sqrt{3}}\geq\frac{4\sqrt{3}}{3R}\Leftrightarrow$$

$$\Leftrightarrow 6R\geq 12r\Leftrightarrow R\geq 2r\ \text{(Euler)}$$

Equality hold for $a=b=c$.

63. In ΔABC the following relationship holds:

$$\frac{a}{h_b^3}+\frac{b}{h_c^3}+\frac{c}{h_a^3}\geq\frac{8\sqrt{3}}{9R^2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a}{h_b^3}+\frac{b}{h_c^3}+\frac{c}{h_a^3}\overset{\text{AM-GM}}{\geq}3\sqrt[3]{\frac{abc}{(h_ah_bh_c)^3}}=$$

$$=\frac{3}{h_ah_bh_c}\cdot\sqrt[3]{4Rrs}\overset{\text{EULER}}{\geq}\frac{3}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}\sqrt[3]{8r^2s}\geq$$

$$\overset{\text{MITRINOVIC}}{\geq}\frac{3abc}{8F^3}\cdot 2\sqrt[3]{r^2\cdot 3\sqrt{3}r}=$$

$$= \frac{3abc}{4F^3} \cdot \sqrt{3} \cdot r = \frac{3\sqrt{3} \cdot 4RFr}{4F^3} = \frac{3\sqrt{3}Rr}{F^2} =$$

$$= \frac{3\sqrt{3}Rr}{r^2 s^2} = \frac{3\sqrt{3}R}{rs^2} \geq \frac{8\sqrt{3}}{9R^2} \Leftrightarrow$$

$$\Leftrightarrow 27\sqrt{3}R^3 \geq 8\sqrt{3}rs^2$$

$$27R^3 = 27R^2 \cdot 2r \geq 8rs^2 \Leftrightarrow 27R^2 \geq 4s^2$$

$$s^2 \leq \frac{27R^2}{4} \Leftrightarrow s \leq \frac{3\sqrt{3}}{2}R \text{ (MITRINOVIC)}$$

Equality holds for $a = b = c$.

64. In ΔABC the following relationship holds:

$$192\sqrt{3}r^3 \leq (a+b)(b+c)(c+a) \leq 24\sqrt{3}R^3$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

We will use the known result:

$$(a+b)(b+c)(c+a) = 2s(s^2 + r^2 + 2Rr)$$

We must prove that:

$$192\sqrt{3}r^3 \leq 2s(s^2 + r^2 + 2Rr) \leq 24\sqrt{3}R^3$$

$$2s(s^2 + r^2 + 2Rr) \overset{\text{MITRINOVIC-GERRETSEN}}{\leq}$$

$$\leq 2 \cdot \frac{3\sqrt{3}R}{2}(4R^2 + 4Rr + 3r^2 + r^2 + 2Rr) =$$

$$= 3\sqrt{3}R(4R^2 + 6Rr + 4r^2) \leq$$

$$\leq 3\sqrt{3}R\left(4R^2 + 6R \cdot \frac{R}{2} + 4 \cdot \frac{R^2}{4}\right) =$$

$$= 3\sqrt{3}R(4R^2 + 4R^2) = 24\sqrt{3}R^3$$

$$2s(s^2 + r^2 + 2Rr) \overset{\text{MITRINOVIC-EULER}}{\geq}$$

$$\geq 2 \cdot 3\sqrt{3}r(27r^2 + r^2 + 2 \cdot 2r \cdot r) =$$

$$= 6\sqrt{3}r(28r^2 + 4r^2) = 6\sqrt{3}r \cdot 32r^2 =$$

$$= 192\sqrt{3}r^3$$

Equality holds for $a = b = c$.

65. In ΔABC the following relationship holds:

$$\frac{a}{r_a \sin^2 A} + \frac{b}{r_b \sin^2 B} + \frac{c}{r_c \sin^2 C} \geq \frac{8\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a}{r_a \sin^2 A} + \frac{b}{r_b \sin^2 B} + \frac{c}{r_c \sin^2 C} = \sum_{cyc} \frac{a}{r_a \sin^2 A} = \sum_{cyc} \frac{a}{\frac{F}{s-a} \cdot \frac{a^2}{4R^2}} =$$

$$= \frac{4R^2}{F} \sum_{cyc} \frac{s-a}{a} = \frac{4R^2}{F}\left(s \sum \frac{1}{a} - 3\right) =$$

$$= \frac{4R^2}{rs}\left(s \cdot \frac{ab+bc+ca}{abc} - 3\right) = \frac{4R^2}{rs} \cdot \left(s \frac{s^2+r^2+4Rr}{4Rrs} - 3\right) =$$

$$= \frac{4R^2}{rs} \cdot \frac{s^2+r^2+4Rr-12Rr}{4Rr} = \frac{R}{r^2 s} \cdot (s^2+r^2-8Rr) \geq$$

$$\overset{\text{GERRETSEN}}{\geq} \frac{R}{r^2 s}(16Rr - 5r^2 + r^2 - 8Rr) =$$

$$= \frac{R}{r^2 s}(8Rr - 4r^2) = \frac{R(8R-4r)}{rs} \overset{\text{EULER}}{\geq}$$

$$\geq \frac{R(16r-4r)}{rs} \overset{\text{MITRINOVIC}}{\geq} \frac{\frac{2}{3\sqrt{3}} s \cdot 12r}{rs} =$$

$$= \frac{24}{3\sqrt{3}} = \frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3}$$

Equality holds for $a = b = c$.

66. In ΔABC the following relationship holds:

$$\frac{a}{r_a \cos^2 \frac{A}{2}} + \frac{b}{r_b \cos^2 \frac{B}{2}} + \frac{c}{r_c \cos^2 \frac{C}{2}} \geq \frac{8\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a}{r_a \cos^2 \frac{A}{2}} + \frac{b}{r_b \cos^2 \frac{B}{2}} + \frac{c}{r_c \cos^2 \frac{C}{2}} \sum_{cyc} \frac{a}{r_a \cos^2 \frac{A}{2}} =$$

$$= \sum_{cyc} \frac{a}{\frac{F}{s-a} \cdot \frac{s(s-a)}{bc}} = \sum_{cyc} \frac{abc}{Fs} =$$

$$= \frac{3abc}{Fs} = \frac{3 \cdot 4RF}{Fs} = \frac{12R}{s} \geq \frac{12 \cdot \frac{2}{3\sqrt{3}} s}{s} = \frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3}$$

Equality holds for $a = b = c$.

Mathematics Department, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania

Email address: dansitaru63@yahoo.com