ROMANIAN MATHEMATICAL MAGAZINE'S CHALLENGES I

DANIEL SITARU - ROMANIA

01. In ΔABC the following relationship holds:

$$\frac{r_a}{b}+\frac{r_b}{c}+\frac{r_c}{a}\geq 3\sqrt{3}\cdot\frac{r}{R}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{r_a}{b}+\frac{r_b}{c}+\frac{r_c}{a}=\sum_{cyc}\frac{r_a}{b}=\sum_{cyc}\frac{F}{b(s-a)}=$$

$$=F\sum_{cyc}\frac{1}{b(s-a)}\overset{\text{AM-GM}}{\geq}\frac{3F}{\sqrt[3]{abc(s-a)(s-b)(s-c)}}=$$

$$=\frac{3F}{\sqrt[3]{4Rrs(s-a)(s-b)(s-c)}}=\frac{3F}{\sqrt[3]{4RrF^2}}=$$

$$=\frac{3F}{\sqrt[3]{4Rrr^2s^2}}=\frac{3F}{r\cdot\sqrt[3]{4Rs^2}}=\frac{3rs}{r\cdot\sqrt[3]{4Rs^2}}\geq$$

$$\overset{\text{MITRINOVIC}}{\geq}\frac{3s}{\sqrt[3]{4R(\frac{3\sqrt{3}R}{2})^2}}\overset{\text{MITRINOVIC}}{\geq}\frac{3\cdot 3\sqrt{3}r}{\sqrt[3]{4R\cdot\frac{27R^2}{4}}}=\frac{9\sqrt{3}r}{3R}=3\sqrt{3}\cdot\frac{r}{R}$$

Equality holds for $a=b=c$.

02. In ΔABC the following relationship holds:

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\geq\frac{3r}{R}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\overset{\text{JENSEN}}{\geq}3\sin\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}=3\sin\frac{\pi}{6}=3\cdot\frac{1}{2}\geq 3\cdot\frac{r}{R}$$

Equality holds for $A=B=C$.

03. In ΔABC the following relationship holds:

$$\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=2$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=$$

$$=\sum_{cyc}\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}=\sum_{cyc}\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{ac\cdot ab}{s(s-b)\cdot s(s-c)}}=$$

$$=\sum_{cyc}\sqrt{\frac{a^2}{s^2}}=\frac{1}{s}\sum_{cyc}a=\frac{a+b+c}{s}=\frac{2s}{s}=2$$

04. In ΔABC the following relationship holds:

$$\sin^2 A\sin 2B+\sin^2 B\sin 2A=2\sin A\sin B\sin C$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sin^2 A\sin 2B+\sin^2 B\sin 2A=\sin^2 A\cdot 2\sin B\cos B+\sin^2 B\cdot 2\sin A\cos A=$$

$$=2\sin A\sin B(\sin A\cos B+\sin B\cos A)=2\sin A\sin B\cdot\sin(A+B)=$$

$$=2\sin A\sin B\cdot\sin(\pi-C)=2\sin A\sin B\sin C$$

05. In ΔABC the following relationship holds:

$$\sin A+\sin B+\sin C\geq 3\sqrt{3}\cdot\frac{r}{R}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sin A+\sin B+\sin C=\sum_{cyc}\sin A=\sum_{cyc}\frac{a}{2R}=\frac{1}{2R}(a+b+c)=\frac{2s}{2R}=\frac{s}{R}\overset{\text{MITRINOVIC}}{\geq}\frac{3\sqrt{3}r}{R}$$

Equality holds for $a=b=c$.

06. In ΔABC the following relationship holds:

$$\cot A+\cot B+\cot C=\frac{2(\sin^2 A+\sin^2 B+\sin^2 C)}{\sin 2A+\sin 2B+\sin 2C}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sin 2A+\sin 2B+\sin 2C=2\sin\frac{2A+2B}{2}\cos\frac{2A-2B}{2}+\sin 2C=$$

$$=2\sin(A+B)\cos(A-B)+2\sin C\cos C=$$

$$=2\sin(\pi-C)\cos(A-B)+2\sin C\cos C=2\sin C(\cos(A-B)+\cos C)=$$

$$=2\sin C\cdot 2\cos\frac{A-B+C}{2}\cos\frac{A-B-C}{2}=$$

$$=4\sin C\cos\frac{\pi-2B}{2}\cos\frac{2A-\pi}{2}=4\sin C\cos\left(\frac{\pi}{2}-B\right)\cos(\frac{\pi}{2}-A\Big)=$$

$$=4\sin C\sin B\sin A$$

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C \tag{1}$$

$$\cot A+\cot B+\cot C=\sum_{cyc}\cot A=\sum_{cyc}\frac{\cos A}{\sin A}=\sum_{cyc}\frac{b^2+c^2-a^2}{2bc\sin A}=$$
$$=\frac{1}{4F}\sum_{cyc}(b^2+c^2-a^2)=\frac{1}{4F}\sum_{cyc}a^2=\frac{1}{4F}\sum_{cyc}4R^2\sin^2 A=$$
$$=\frac{4R^2(\sin^2 A+\sin^2 B+\sin^2 C)}{4\cdot 2R^2\sin A\sin B\sin C}=\frac{2(\sin^2 A+\sin^2 B+\sin^2 C)}{4\sin A\sin B\sin C}\overset{(1)}{=}$$
$$=\frac{2(\sin^2 A+\sin^2 B+\sin^2 C)}{\sin 2A+\sin 2B+\sin 2C}$$

07. In ΔABC the following relationship holds:
$$h_a h_b h_c\geq 27r^3$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$h_a h_b h_c=\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}=\frac{8F^3}{abc}=\frac{8F^3}{4RF}=\frac{2F^2}{R}=\frac{2r^2s^2}{R}$$

We must prove that:
$$\frac{2r^2s^2}{R}\geq 27r^3\Leftrightarrow\frac{2s^2}{R}\geq 27r\Leftrightarrow 27r\Leftrightarrow 2s^2\geq 27Rr$$
$$2s^2\overset{\text{GERRETSEN}}{\geq}2(16Rr-5r^2)\geq 27Rr$$
$$32Rr-10r^2\geq 27Rr$$
$$5Rr\geq 10r^2$$
$$R\geq 2R\ (\text{Euler})$$

Equality holds for $a=b=c$.

08. In ΔABC the following relationship holds:
$$\frac{h_a+h_b}{a}+\frac{h_b+h_c}{b}+\frac{h_c+h_a}{c}\geq 6\sqrt{3}\cdot\frac{r}{R}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

Lemma: In ΔABC holds:
$$h_a h_b h_c\geq 27r^3$$

Proof.
$$h_a h_b h_c=\frac{8F^3}{abc}=\frac{8F^3}{4RF}=\frac{2F^2}{R}=\frac{2r^2s^2}{R}$$
$$\frac{2r^2s^2}{R}\geq 27r^3\Leftrightarrow\frac{2s^2}{R}\geq 27r\Leftrightarrow 2s^2\geq 27Rr$$
$$2s^2\overset{\text{GERRETSEN}}{\geq}2(16Rr-5r^2)\geq 27Rr$$
$$32Rr-10r^2\geq 27Rr\Leftrightarrow R\geq 2r\ (\text{Euler})$$

□

Back to the problem:

$$\sum_{cyc}\frac{h_a+h_b}{a}\overset{\text{AM-GM}}{\geq}3\sqrt[3]{\frac{(h_a+h_b)(h_b+h_c)(h_c+h_a)}{abc}}\geq$$

$$\overset{\text{CESARO}}{\geq}3\sqrt[3]{\frac{8h_ah_bh_c}{4RF}}\overset{\text{Lemma}}{\geq}3\sqrt[3]{\frac{8\cdot 27r^3}{4Rrs}}=3\cdot 2\cdot 3r\cdot\sqrt[3]{\frac{1}{4rs}}=\frac{18r}{\sqrt[3]{2R^2s}}\overset{\text{EULER}}{\geq}$$

$$\geq\frac{18r}{\sqrt[3]{4R\cdot\frac{R}{2}\cdot s}}=\frac{18r}{\sqrt[3]{2R^2s}}\overset{\text{MITRINOVIC}}{\geq}$$

$$\geq\frac{18r}{\sqrt[3]{2R^2\cdot\frac{3\sqrt{3}}{2}\cdot R}}=\frac{18r}{\sqrt[3]{(R\sqrt{3})^3}}=\frac{18r}{\sqrt{3}R}=\frac{18\sqrt{3}r}{3R}=6\sqrt{3}\cdot\frac{r}{R}$$

Equality holds for $a=b=c$.

09. In ΔABC the following relationship holds:

$$\frac{AI\cdot BI\cdot CI}{abc}\leq\frac{1}{3\sqrt{3}}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{AI\cdot BI\cdot CI}{abc}=\frac{\frac{r}{\sin\frac{A}{2}}\cdot\frac{r}{\sin\frac{B}{2}}\cdot\frac{r}{\sin\frac{C}{2}}}{abc}=\frac{r^3}{abc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=$$

$$=\frac{r^3}{abc\sqrt{\frac{(s-b)(s-c)(s-a)(s-c)(s-a)(s-b)}{bc\cdot ac\cdot ab}}}=$$

$$=\frac{r^3}{abc\cdot\frac{(s-a)(s-b)(s-c)}{abc}}=\frac{r^3s}{s(s-a)(s-b)(s-c)}=$$

$$=\frac{r^3s}{F^2}=\frac{r^3s}{r^2s^2}=\frac{r}{s}\overset{\text{MITRINOVIC}}{\leq}\frac{r}{3\sqrt{3}r}=\frac{1}{3\sqrt{3}}$$

Equality holds for $a=b=c$.

10. In ΔABC the following relationship holds:

$$\frac{a+b}{h_a}+\frac{b+c}{h_b}+\frac{c+a}{h_c}\geq 4\sqrt{3}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a+b}{h_a}+\frac{b+c}{h_b}+\frac{c+a}{h_c}=\sum_{cyc}\frac{a+b}{h_a}\overset{\text{AM-GM}}{\geq}$$

$$\geq 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{h_ah_bh_c}}\overset{\text{CESARO}}{\geq}3\sqrt[3]{\frac{8abc}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}}=$$

$$=3\sqrt[3]{\frac{(abc)^2}{F^3}}=3\sqrt[3]{\frac{16R^2F^2}{F^3}}=3\sqrt[3]{\frac{16R^2}{F}}\geq$$

$$\overset{\text{MITRINOVIC}}{\geq} 3\sqrt[3]{\frac{16\cdot\frac{4}{27}s^2}{rs}} = 3\cdot\frac{4}{3}\sqrt[3]{\frac{s}{r}} \overset{\text{MITRINOVIC}}{\geq} 4\sqrt[3]{\frac{3\sqrt{3}r}{r}} = 4\sqrt[3]{(\sqrt{3})^3} = 4\sqrt{3}$$

Equality holds for $a = b = c$.

11. In ΔABC the following relationship holds:

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \geq 3\sqrt{3}\cdot\frac{r}{R}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \overset{\text{AM-GM}}{\geq} 3\sqrt[3]{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} =$$

$$= 3\sqrt[3]{\sqrt{\frac{s(s-a)\cdot s(s-b)\cdot s(s-c)}{bc\cdot ac\cdot ab}}} = 3\sqrt[6]{\frac{s^3(s-a)(s-b)(s-c)}{(abc)^2}} = 3\sqrt[6]{\frac{s^2F^2}{(abc)^2}} =$$

$$= 3\sqrt[3]{\frac{sF}{abc}} = 3\sqrt[3]{\frac{sF}{4RF}} = 3\sqrt[3]{\frac{s}{4R}}$$

Remains to prove:

$$3\sqrt[3]{\frac{s}{4R}} \geq 3\sqrt{3}\cdot\frac{r}{R} \Leftrightarrow \frac{s}{4R} \geq 3\sqrt{3}\cdot\frac{r^3}{R^3}$$

$$sR^3 \geq 12\sqrt{3}Rr^3 \Leftrightarrow sR^2 \geq 12\sqrt{3}r^3 \text{ (to prove)}$$

$$sR^2 \overset{\text{EULER}}{\geq} s\cdot(2r)^2 \overset{\text{MITRINOVIC}}{\geq} 3\sqrt{3}r\cdot 4r^2 = 12\sqrt{3}r^3$$

Equality holds for $A = B = C$.

12. If $a, b > 0$ and $a + b = 2ab$ then:

$$\frac{a}{b^2} + \frac{b}{a^2} \geq 2$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a}{b^2} + \frac{b}{a^2} \geq 2 \Leftrightarrow a^3 + b^3 \geq 2a^2b^2, \qquad (a+b)(a^2 - ab + b^2) \geq 2a^2b^2$$

$$2ab(a^2 - ab + b^2) \geq 2a^2b^2, \qquad a^2 - ab + b^2 \geq ab$$

$$a^2 - 2ab + b^2 \geq 0, \qquad (a-b)^2 \geq 0$$

Equality holds for $a = b$.

13. If $a, b > 0; a + b = 2a^2b^2$ then:

$$\frac{a+1}{b^2} + \frac{b+1}{a^2} \geq 4$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania
Denote: $S = a + b; P = ab; S > 0; P > 0$

$$a + b \overset{\text{AM-GM}}{\geq} 2\sqrt{ab} \Rightarrow S \geq 2\sqrt{P} \Rightarrow 2P^2 \geq 2\sqrt{P}$$
$$\Rightarrow P^2 \geq P \Rightarrow P^4 \geq P \Rightarrow P^3 \geq 1 \Rightarrow P \geq 1$$
$$\Rightarrow P - 1 \geq 0 \tag{1}$$

We used the hypothesis: $S = 2P^2$.

$$\frac{a+1}{b^2} + \frac{b+1}{a^2} \geq 4 \Leftrightarrow a^2(a+1) + b^2(b+1) \geq 4a^2b^2$$
$$a^3 + b^3 + a^2 + b^2 - 4a^2b^2 \geq 0$$
$$S^3 - 3SP + S^2 - 2P - 4P^2 \geq 0$$
$$8P^6 - 6P^3 + 4P^4 - 2P - 4P^2 \geq 0$$
$$4P^5 - 3P^2 + 2P^3 - 1 - 2P \geq 0$$
$$4P^5 + 2P^3 - 3P^2 - 2P - 1 \geq 0$$
$$4P^5 - 4P^4 + 4P^4 - 4P^3 + 6P^3 - 6P^2 + 3P^2 - 3P + P - 1 \geq 0$$
$$4P^4(P-1) + 4P^3(P-1) + 6P^2(P-1) + 3P(P-1) + (P-1) \geq 0$$
$$(P-1)(4P^4 + 4P^3 + 6P^2 + 3P + 1) \geq 0$$
$$P - 1 \geq 0 \text{ (True by (1))}$$

Equality holds for $a = b = 1$.

14. If $a, b > 0; ab = a + b$ then:

$$\frac{a+1}{b^3} + \frac{b+1}{a^3} \geq \frac{3}{4}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania
Denote: $S = a + b; P = ab; S > 0; P > 0$

$$a + b \overset{AM-GM}{\geq} 2\sqrt{ab} \Rightarrow S \geq 2\sqrt{P} \Rightarrow S \geq 2\sqrt{S} \Rightarrow$$
$$\Rightarrow S^2 \geq 4S \Rightarrow S \geq 4 \Rightarrow S - 4 \geq 0 \tag{1}$$
$$\frac{a+1}{b^3} + \frac{b+1}{a^3} \geq \frac{3}{4} \Leftrightarrow 4[a^3(a+1) + b^3(b+1)] \geq 3a^3b^3$$
$$4(a^4 + b^4) + 4(a^3 + b^3) \geq 3a^3b^3$$
$$4[(S^2 - 2P)^2 - 2P^2] + 4(S^3 - 3SP) - 3P^3 \geq 0$$
$$S = P \Rightarrow$$
$$\Rightarrow 4(S^2 - 2S)^2 - 8S^2 + 4S^3 - 12S^2 - 3S^3 \geq 0$$
$$4S^4 - 16S^3 + 16S^2 - 8S^2 + S^3 - 12S^2 \geq 0$$
$$4S^4 - 15S^3 - 4S^2 \geq 0 \Leftrightarrow 4S^2 - 15S - 4 \geq 0$$
$$4S^2 - 16S + S - 4 \geq 0 \Leftrightarrow 4S(S-4) + (S-4) \geq 0$$
$$(S-4)(4S+1) \geq 0 \Leftrightarrow S - 4 \text{ (True by (1))}$$

Equality holds for $a = b = 2$.

15. In ΔABC the following relationship holds:

$$\frac{a+b}{\sin^2 C}+\frac{b+c}{\sin^2 A}+\frac{c+a}{\sin^2 B}\geq 8\sqrt{3}R$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a+b}{\sin^2 C}+\frac{b+c}{\sin^2 A}+\frac{c+a}{\sin^2 B}=\sum_{cyc}\frac{a+b}{\sin^2 C}\geq$$

$$\geq 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{\sin^2 A\sin^2 B\sin^2 C}}\overset{\text{CESARO}}{\geq}3\sqrt[3]{\frac{8abc}{\frac{a^2}{4R^2}\cdot\frac{b^2}{4R^2}\cdot\frac{c^2}{4R^2}}}=3\cdot 4R^2\sqrt[3]{\frac{8abc}{a^2b^2c^2}}=$$

$$=12R^2\cdot 2\cdot\frac{1}{\sqrt[3]{abc}}=\frac{24R^2}{\sqrt[3]{4Rrs}}\overset{\text{EULER}}{\geq}\frac{24R^2}{\sqrt[3]{4R\cdot\frac{R}{2}\cdot s}}=\frac{24R^2}{\sqrt[3]{2R^2\cdot\frac{3\sqrt{3}}{2}R}}=$$

$$=\frac{24R^2}{\sqrt[3]{(\sqrt{3}R)^3}}=\frac{24R^2}{\sqrt{3}R}=\frac{24R}{\sqrt{3}}=\frac{24\sqrt{3}R}{3}=8\sqrt{3}R$$

16. In ΔABC the following relationship holds:

$$\frac{a+b}{(\sin A+\sin B)^2}+\frac{b+c}{(\sin B+\sin C)^2}+\frac{c+a}{(\sin C+\sin A)^2}\geq 2\sqrt{3}R$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a+b}{(\sin A+\sin B)^2}+\frac{b+c}{(\sin B+\sin C)^2}+\frac{c+a}{(\sin C+\sin A)^2}=$$

$$=\sum_{cyc}\frac{a+b}{(\sin A+\sin B)^2}=\sum_{cyc}\frac{a+b}{\left(\frac{a}{2R}+\frac{b}{2R}\right)^2}=4R^2\sum_{cyc}\frac{a+b}{(a+b)^2}=$$

$$=4R^2\cdot\sum_{cyc}\frac{1}{a+b}\overset{\text{BERGSTROM}}{\geq}4R^2\cdot\frac{(1+1+1)^2}{a+b+b+c+c+a}=$$

$$=4R^2\cdot\frac{9}{4s}=\frac{9R^2}{s}\overset{\text{MITRINOVIC}}{\geq}\frac{9R^2}{3\sqrt{3}}=\frac{3R\cdot 2}{\sqrt{3}}=2\sqrt{3}R$$

Equality holds for $a=b=c$.

17. In ΔABC the following relationship holds:

$$\frac{a+b}{(\sin A+\sin B)^3}+\frac{b+c}{(\sin B+\sin C)^3}+\frac{c+a}{(\sin C+\sin A)^3}\geq 2R$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a+b}{(\sin A+\sin B)^3}+\frac{b+c}{(\sin B+\sin C)^3}+\frac{c+a}{(\sin C+\sin A)^3}-$$

$$=\sum_{cyc}\frac{a+b}{(\sin A+\sin B)^3}=\sum_{cyc}\frac{a+b}{(\frac{a}{2R}+\frac{b}{2R})^3}=$$

$$= 8R^3 \sum_{cyc} \frac{a+b}{(a+b)^3} = 8R^3 \sum_{cyc} \frac{1}{(a+b)^2} \overset{\text{RADON}}{\geq}$$

$$\geq 8R^3 \cdot \frac{(1+1+1)^3}{(a+b+b+c+c+a)^2} = \frac{8R^3}{4} \cdot \frac{27}{4s^2} =$$

$$= \frac{27R^3}{2s^2} \overset{\text{MITRINOVIC}}{\geq} \frac{27R^3}{2 \cdot (\frac{3\sqrt{3}}{2}R)^2} = \frac{27R^3}{2 \cdot \frac{27R^2}{4}} = 2R$$

Equality holds for $a = b = c$.

18. If $a, b, c > 0$ then:

$$\sum_{cyc} \frac{\sqrt{(1+b^2)(1+c^2)}}{a} \geq 6$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sum_{cyc} \frac{\sqrt{(1+b^2)(1+c^2)}}{a} \overset{\text{AM-GM}}{\geq} \sum_{cyc} \frac{\sqrt{2b \cdot 2c}}{a} =$$

$$= 2\sum_{cyc} \frac{\sqrt{bc}}{a} \overset{\text{AM-GM}}{\geq} 2 \cdot 3\sqrt[3]{\frac{\sqrt{bc}}{a} \cdot \frac{\sqrt{ca}}{b} \cdot \frac{\sqrt{ab}}{c}} = 6\sqrt[3]{\frac{abc}{abc}} = 6 \cdot 1 = 6$$

Equality holds for $a = b = c = 1$.

19. If $a, b > 0, ab = 1$ then:

$$a^{a^2+2b^2} \cdot b^{b^2+2a^2} \leq 1$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$a^{a^2+2b^2} \cdot b^{b^2+2a^2} \leq 1 \Rightarrow \ln(a^{a^2+2b^2} \cdot b^{b^2+2a^2}) \leq 1$$

$$(a^2+2b^2)\ln a + (b^2+2a^2)\ln b \leq 0$$

$$\left(a^2 + \frac{2}{a^2}\right)\ln a + \left(\frac{1}{a^2} + 2a^2\right)\ln\frac{1}{a} \leq 0$$

$$a^2 \ln a + \frac{2}{a^2}\ln a + \frac{1}{a^2}\ln a^{-1} + 2a^2 \ln a^{-1} \leq 0$$

$$a^2 \ln a + \frac{2}{a^2}\ln a - \frac{1}{a^2}\ln a - 2a^2 \ln a \leq 0$$

$$\left(a^2 + \frac{2}{a^2} - \frac{1}{a^2} - 2a^2\right)\ln a \leq 0$$

$$\left(\frac{1}{a^2} - a^2\right)\ln a \leq 0 \Leftrightarrow \frac{a^4-1}{a^2}\ln a \geq 0$$

$$\frac{(a-1)(a+1)(a^2+1)}{a^2}\ln a \geq 0 \Leftrightarrow (a-1)\ln a \geq 0$$

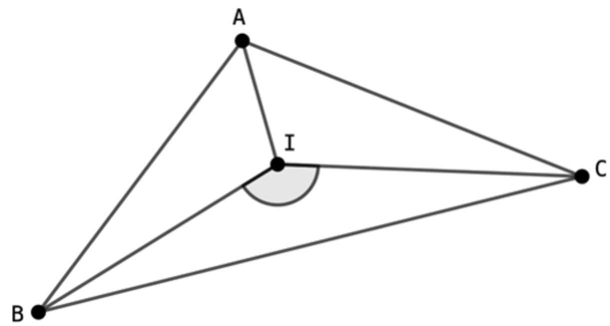
If $a \geq 1 \Rightarrow a - 1 \geq 0; \ln a \geq 0 \Rightarrow (a-1)\ln a \geq 0$
If $0 < a \leq 1 \Rightarrow a - 1 \leq 0; \ln a \leq 0 \Rightarrow (a-1)\ln a \geq 0$
Equality holds for $a = b = 1$.

20. If $\Delta ABC, I-$ incenter, the following relationship holds:

$$\sin A+\sin B+\sin C=4\sin(\widehat{AIB})\sin(\widehat{BIC})\sin(\widehat{CIA})$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania



$$\angle BIC=180^\circ-\frac{\widehat{B}}{2}-\frac{\widehat{C}}{2}=180^\circ-\frac{\widehat{B}+\widehat{C}}{2}=$$

$$=180^\circ-\frac{180^\circ-\widehat{A}}{2}=180^\circ-90^\circ+\frac{\widehat{A}}{2}=90^\circ+\frac{\widehat{A}}{2}$$

$$\sin(\widehat{BIC})=\sin\left(90^\circ+\frac{A}{2}\right)=\sin 90^\circ\cos\frac{A}{2}\cos 90^\circ=\cos\frac{A}{2}$$

$$\text{Analogous: } \sin(\widehat{AIB})=\cos\frac{C}{2};\sin(\widehat{CIA})=\sin\frac{B}{2}$$

$$\sin A+\sin B+\sin C=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+\sin\left(2\cdot\frac{C}{2}\right)=$$

$$=2\sin\frac{180^\circ-C}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}=2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\cos\frac{C}{2}\cdot\cos\left(90^\circ-\frac{C}{2}\right)$$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{180^\circ-C}{2}\right)=$$

$$=2\cos\frac{C}{2}\cdot 2\cos\frac{A-B+180^\circ-C}{4}\cos\frac{A-B-180^\circ+C}{4}=$$

$$=4\cos\frac{C}{2}\cos\frac{A-B-C+A+B+C}{4}\cos\frac{A-B-A-B-C+C}{4}=$$

$$=4\cos\frac{C}{2}\cos\frac{2A}{4}\cos\left(-\frac{2B}{4}\right)=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=4\sin(\widehat{AIB})\sin(\widehat{BIC})\sin(\widehat{CIA})$$

21. In ΔABC the following relationship holds:

$$\sin 2026A+\sin 2026B+\sin 2026C=4\sin 1013A\sin 1013B\sin 1013C$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

Denote:

$$1013A=x;1013B=y;1013C=z$$

$$x+y+z=1013(A+B+C)=1013\pi \tag{1}$$

$$\cos\left(\frac{1013\pi}{2}-y\right)=\cos\frac{1013\pi}{2}\cos y+\sin\frac{1013\pi}{2}\sin y=$$

$$= \cos\left(506\pi + \frac{\pi}{2}\right)\cos y + \sin\left(506\pi + \frac{\pi}{2}\right)\sin y = \cos\frac{\pi}{2}\cos y + \sin\frac{\pi}{2}\sin y = \sin y$$

$$\cos\left(\frac{1013\pi}{2} - y\right) = \sin y \tag{2}$$

Analogous:

$$\cos\left(x - \frac{1013\pi}{2}\right) = \cos\left(\frac{1013\pi}{2} - c\right) = \sin x \tag{3}$$

$$\sin 2026A + \sin 2026B + \sin 2026C = \sin 2x + \sin 2y + \sin 2z =$$
$$= 2\sin\frac{2x+2y}{2}\cos\frac{2x-2y}{2} + 2\sin z\cos z = 2\sin(x+y)\cos(x-y) + 2\sin z\cos z \overset{(1)}{=}$$
$$= 2\sin(1013\pi - z)\cos(x-y) + 2\sin z\cos z =$$
$$= 2(\sin 1013\pi\cos z - \sin z\cos 1013\pi)\cos(x-y) + 2\sin z\cos z =$$
$$= 2(0\cdot\cos z - \sin z\cdot(-1)^{1013})\cos(x-y) + 2\sin z\cos z =$$
$$= 2\sin z\cos(x-y) + 2\sin z\cos z = 2\sin z(\cos(x-y) + \cos z) =$$
$$= 2\sin z\cdot 2\cos\frac{x-y+z}{2}\cdot\cos\frac{x-y-z}{2} = 4\sin z\cos\frac{x+z-y}{2}\cos\frac{y+z-x}{2} \overset{(1)}{=}$$
$$= 4\sin z\cos\left(\frac{1013\pi - 2y}{2}\right)\cos\left(\frac{1013\pi - 2x}{2}\right) \overset{(2),(3)}{=} 4\sin z\sin y\sin x =$$
$$= 4\sin 1013A\sin 1013B\sin 1013C$$

22. If in ΔABC we have: $\tan\frac{A}{2}\tan\frac{B}{2} = \frac{1}{3}$ then:

$$2\sin C = \sin A + \sin B$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\tan\frac{A}{2}\tan\frac{B}{2} = \frac{1}{3} \Rightarrow \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\cdot\sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \frac{1}{3}$$

$$\Rightarrow \frac{s-c}{s} = \frac{1}{3} \Rightarrow 3s - 3c = s \Rightarrow 2s - 3c = 0 \Rightarrow a + b + c - 3c = 0 \Rightarrow 2c = a + b$$
$$2\cdot 2R\sin C = 2R\sin A + 2R\sin B, \qquad 2\sin C = \sin A + \sin B$$

23. In ΔABC the following relationship holds:

$$\cot A + \cot B + \cot C = \frac{\sin^2 A + \sin^2 B + \sin^2 C}{2\sin A\sin B\sin C}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\cot A + \cot B + \cot C = \sum_{cyc}\cot A = \sum_{cyc}\frac{\cos A}{\sin A} =$$

$$= \sum_{cyc}\frac{\frac{b^2+c^2-a^2}{2bc}}{\sin A} = \sum_{cyc}\frac{b^2+c^2-a^2}{2bc\sin A} = \sum_{cyc}\frac{b^2+c^2-a^2}{4F} =$$

$$= \frac{1}{4F}\left(\sum_{cyc}b^2 + \sum_{cyc}c^2 - \sum_{cyc}a^2\right) = \frac{1}{4F}\left(\sum_{cyc}a^2 + \sum_{cyc}a^2 - \sum_{cyc}a^2\right) = \frac{1}{4F}\sum_{cyc}a^2 =$$

$$= \frac{1}{4F}(a^2+b^2+c^2) = \frac{1}{4 \cdot 2R^2 \sin A \sin B \sin C} \sum_{cyc} 4R^2 \sin^2 A =$$

$$= \frac{4R^2}{8R^2} \cdot \frac{\sin^2 A + \sin^2 B + \sin^2 C}{\sin A \sin B \sin C} = \frac{\sin^2 A + \sin^2 B + \sin^2 C}{2 \sin A \sin B \sin C}$$

24. In ΔABC the following relationship holds:

$$\frac{a}{\sin^3 B} + \frac{b}{\sin^3 C} + \frac{c}{\sin^3 A} \geq 8R$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a}{\sin^3 B} + \frac{b}{\sin^3 C} + \frac{c}{\sin^3 A} = \sum_{cyc} \frac{a}{\sin^3 B} \overset{\text{AM-GM}}{\geq} 3 \cdot \sqrt[3]{\frac{abc}{\sin^3 A \sin^3 B \sin^3 C}} =$$

$$= 3\sqrt[3]{\frac{abc}{\left(\frac{abc}{2R}\right)^3}} = 3\sqrt[3]{\frac{(8R^3)^3}{(abc)^2}} = \frac{24R^3}{\sqrt[3]{(abc)^2}} = \frac{24R^3}{\sqrt[3]{16R^2r^2s^2}} \overset{\text{EULER}}{\geq} \frac{24R^3}{\sqrt[3]{16R^2 \cdot \frac{R^2}{4} s^2}} \overset{\text{MITRINOVIC}}{\geq}$$

$$\geq \frac{24R^3}{\sqrt[3]{4R^4 \cdot \frac{27R^2}{4}}} = \frac{24R^3}{3R^2} = 8R$$

Equality holds for $a = b = c$.

25. In ΔABC the following relationship holds:

$$\frac{h_a}{b} + \frac{h_b}{c} + \frac{h_c}{a} \leq \frac{3\sqrt{3}}{2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{h_a}{b} + \frac{h_b}{c} + \frac{h_c}{a} = \sum_{cyc} \frac{h_a}{b} = \sum_{cyc} \frac{\frac{2F}{a}}{b} = 2F \sum_{cyc} \frac{1}{ab} = 2F \cdot \frac{a+b+c}{abc} =$$

$$= 2F \cdot \frac{2s}{4RF} = \frac{s}{R} \overset{\text{MITRINOVIC}}{\leq} \frac{1}{R} \cdot \frac{3\sqrt{3}}{2} R = \frac{3\sqrt{3}}{2}$$

Equality holds for $a = b = c$.

26. In ΔABC the following relationship holds:

$$\frac{a+b}{\sin A} + \frac{b+c}{\sin B} + \frac{c+a}{\sin C} \geq 12R$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a+b}{\sin A} + \frac{b+c}{\sin B} + \frac{c+a}{\sin C} = \sum_{cyc} \frac{a+b}{\sin A} = \sum_{cyc} \frac{a+b}{\frac{a}{2R}} = 2R \sum_{cyc} \frac{a+b}{a} \overset{\text{AM-GM}}{\geq}$$

$$\geq 2R \cdot 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}} \overset{\text{CESARP}}{\geq} 6R \cdot \sqrt[3]{\frac{8abc}{abc}} = 6R\sqrt[3]{8} = 6R \cdot 2 = 12R$$

27. In ΔABC the following relationship holds:

$$\frac{h_a}{b}+\frac{h_b}{c}+\frac{h_c}{a}\le\frac{3\sqrt{3}}{4}\cdot\frac{R}{r}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{h_a}{b}+\frac{h_b}{c}+\frac{h_c}{a}=\sum_{cyc}\frac{h_a}{b}=\sum_{cyc}\frac{\frac{2F}{a}}{b}=2F\sum_{cyc}\frac{1}{ab}=2F\sum_{cyc}\frac{c}{abc}=$$

$$=\frac{2F}{abc}\sum_{cyc}c=\frac{2F}{4RF}(a+b+c)=\frac{1}{2R}\cdot 2s=\frac{s}{r}\overset{\text{EULER}}{\le}\frac{s}{2R}\overset{\text{MITRINOVIC}}{\le}\frac{3\sqrt{3}}{2}R\cdot\frac{1}{2R}=\frac{3\sqrt{3}}{4}\cdot\frac{R}{r}$$

Equality holds for $a=b=c$.

28. In ΔABC the following relationship holds:

$$r_ar_b+r_br_c+r_cr_a\ge h_ah_b+h_bh_c+h_ch_a$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sum_{cyc}r_ar_b\ge\sum_{cyc}h_ah_b$$

$$\sum_{cyc}\frac{F}{s-a}\cdot\frac{F}{s-b}\ge\sum_{cyc}\frac{2F}{a}\cdot\frac{2F}{b}$$

$$\sum_{cyc}\frac{1}{(s-a)(s-b)}\ge 4\sum_{cyc}\frac{1}{ab},\qquad\frac{s-c+s-b+s-a}{(s-a)(s-b)(s-c)}\ge 4\cdot\frac{a+b+c}{abc}$$

$$\frac{s}{(s-a)(s-b)(s-c)}\ge 4\cdot\frac{2s}{4RF},\qquad\frac{s^2}{s(s-a)(s-b)(s-c)}\ge\frac{2s}{R\cdot rs}$$

$$\frac{s^2}{F^2}\ge\frac{2}{Rr},\qquad\frac{s^2}{r^2s^2}\ge\frac{2}{Rr},\qquad\frac{1}{r^2}\ge\frac{2}{Rr}$$

$$\Leftrightarrow R\ge 2r\ \text{(Euler)}$$

Equality holds for $a=b=c$.

29. In ΔABC the following relationship holds:

$$\frac{r_ar_br_c}{h_ah_bh_c}=\frac{R}{2r}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{r_ar_br_c}{h_ah_bh_c}=\frac{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}=\frac{abc}{8(s-a)(s-b)(s-c)}=$$

$$=\frac{abc}{8s(s-a)(s-b)(s-c)}=\frac{4RFs}{8F^2}=\frac{4Rr}{8F}=\frac{4Rs}{8rs}=\frac{R}{2r}$$

30. In ΔABC the following relationship holds:

$$216r^3\le(h_a+h_b)(h_b+h_c)(h_c+h_a)\le 27R^3$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$(h_a+h_b)(h_b+h_c)(h_c+h_a)=\prod_{cyc}(h_a+h_b)=\prod_{cyc}\Big(\frac{2F}{a}+\frac{2F}{b}\Big)=8F^3\prod_{cyc}\Big(\frac{1}{a}+\frac{1}{b}\Big)=$$

$$=8F^3\prod_{cyc}\Big(\frac{a+b}{ab}\Big)=8F^3\cdot\frac{(a+b)(b+c)(c+a)}{abc\cdot abc}\overset{\text{CESARO}}{\geq}$$

$$\geq 8F^3\cdot\frac{8abc}{abc\cdot abc}=\frac{64F^3}{abc}=\frac{64F^3}{4RF}=\frac{16F^2}{R}=\frac{16r^2s^2}{R}$$

Remains to prove:

$$216r^3\leq\frac{16r^2s^2}{R}\leq 27R^3$$

$$\frac{16r^2s^2}{R}\overset{\text{EULER}}{\leq}\frac{16\cdot(\frac{R}{2})^2\cdot s^2}{R}=4Rs^2\overset{\text{MITRINOVIC}}{\leq}4R\cdot\Big(\frac{3\sqrt{3}}{2}R\Big)^2=4R\cdot\frac{27R^2}{4}=27R^3$$

$$\frac{16r^2s^2}{R}\geq 216r^3\Leftrightarrow\frac{2r^2s^2}{R}\geq 27r^3\Leftrightarrow\frac{2s^2}{R}\geq 27r\Leftrightarrow 2s^2\geq 27Rr\ \text{(to prove)}$$

$$2s^2\overset{\text{GERRETSEN}}{\geq}2(16Rr-5r^2)=32Rr-10r^2\geq 27Rr\Leftrightarrow 5Rr\geq 10r^2\Leftrightarrow R\geq 2r$$

(Euler)

Equality holds for $a=b=c$.

31. In ΔABC the following relationship holds:

$$216r^3\leq(r_a+r_b)(r_b+r_c)(r_c+r_a)\leq 27R^3$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$(r_a+r_b)(r_b+r_c)(r_c+r_a)=\prod_{cyc}(r_a+r_b)=$$

$$=\prod_{cyc}\Big(\frac{F}{s-a}+\frac{F}{s-b}\Big)=F^3\prod_{cyc}\Big(\frac{1}{s-a}+\frac{1}{s-b}\Big)=$$

$$=F^3\prod_{cyc}\frac{s-b+s-a}{(s-a)(s-b)}=F^3\prod_{cyc}\frac{2s-a-b}{(s-a)(s-b)}=$$

$$=F^3\prod_{cyc}\frac{a+b+c-a-b}{(s-a)(s-b)}=F^3\prod_{cyc}\frac{c}{(s-a)(s-b)}=$$

$$=F^3\cdot\frac{abc}{(s-a)^2(s-b)^2(s-c)^2}=\frac{F^3\cdot abc\cdot s^2}{s^2(s-a)^2(s-b)^2(s-c)^2}=\frac{F^3\cdot 4RF\cdot s^2}{F^4}=4Rs^2$$

Remains to prove:

$$216r^3\leq 4Rs^2\leq 27R^3$$

$$4Rs^2\overset{\text{MITRINOVIC}}{\leq}4R\cdot\Big(\frac{3\sqrt{3}}{2}R\Big)^2=4R\cdot\frac{27R^2}{4}=27R^3$$

$$4Rs^2\overset{\text{MITRINOVIC}}{\geq}4R\cdot(3\sqrt{3}r)^2=4R\cdot 27r^2\overset{\text{EULER}}{\geq}4\cdot 2r\cdot 27r^2=216r^3$$

Equality holds for $a=b=c$.

32. In ΔABC the following relationship holds:

$$(h_a+h_b)^{\cos\frac{C}{2}}\cdot(h_b+h_c)^{\cos\frac{A}{2}}\cdot(h_c+h_a)^{\cos\frac{B}{2}}\le(3R)^{\frac{3\sqrt{3}}{2}}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\text{WLOG: } a\le b\le c\Rightarrow\frac{1}{a}\ge\frac{1}{b}\ge\frac{1}{c}\Rightarrow\frac{2F}{a}\ge\frac{2F}{b}\ge\frac{2F}{c}\Rightarrow$$

$$\Rightarrow h_a\ge h_b\ge h_c\Rightarrow\begin{cases}h_a+h_c\ge h_b+h_c\\ h_b+h_a\ge h_c+h_a\end{cases}\Rightarrow$$

(1) $$\Rightarrow h_a+h_b\ge h_a+h_c\ge h_b+h_c$$

(2) $$a\le b\le c\Rightarrow\cos\frac{A}{2}\ge\cos\frac{B}{2}\cos\frac{C}{2}\Rightarrow\cos\frac{C}{2}\le\cos\frac{B}{2}\le\cos\frac{A}{2}$$

By (1); (2) and Cebyshev's inequality:

$$(h_a+h_b)\cos\frac{C}{2}+(h_b+h_c)\cos\frac{A}{2}+(h_c+h_a)\cos\frac{A}{2}\le$$

$$\le\frac{1}{3}(h_a+h_b+h_b+h_c+h_c+h_a)\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)=$$

(3) $$=\frac{2}{3}(h_a+h_b+h_c)\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)$$

$$\prod_{cyc}(h_a+h_b)\cos\frac{C}{2}\overset{\text{WEIGHTED AM-GM}}{\le}$$

$$\le\left(\frac{(h_a+h_b)\cos\frac{C}{2}+(h_b+h_c)\cos\frac{A}{2}+(h_c+h_a)\cos\frac{B}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\right)^{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\le$$

$$\overset{(3)}{\le}\left(\frac{\frac{2}{3}(h_a+h_b+h_c)(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2})}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\right)^{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\le$$

$$\overset{\text{Jensen}}{\le}\left(\frac{2}{3}\left(\frac{2F}{a}+\frac{2F}{b}+\frac{2F}{c}\right)\right)^{3\cos\frac{A+B+C}{6}}=\left(\frac{4F}{3}\cdot\frac{ab+bc+ca}{abc}\right)^{3\cos\frac{\pi}{6}}=$$

$$=\left(\frac{4F}{3}\cdot\frac{s^2+r^2+4Rr}{4RF}\right)^{\frac{3\sqrt{3}}{2}}=\left(\frac{s^2+r^2+4Rr}{3R}\right)^{\frac{3\sqrt{3}}{2}}\overset{\text{MITRINOVIC}}{\le}\left(\frac{\frac{27R^2}{4}+r^2+4Rr}{3R}\right)^{\frac{3\sqrt{3}}{2}}\overset{\text{EULER}}{\le}$$

$$\le\left(\frac{\frac{27R^2}{4}+\frac{R^2}{4}+4R\cdot\frac{R}{2}}{3R}\right)^{\frac{3\sqrt{3}}{2}}=\left(\frac{7R^2+2R^2}{3R}\right)^{\frac{3\sqrt{3}}{2}}=(3R)^{\frac{3\sqrt{3}}{2}}$$

Equality holds for $a=b=c$.

33. In ΔABC the following relationship holds:

$$(ab)^{\sin C}\cdot(bc)^{\sin A}\cdot(ca)^{\sin B}\le(3R^2)^{\frac{3\sqrt{3}}{2}}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$(ab)^{\sin C}\cdot(bc)^{\sin A}\cdot(ca)^{\sin B}\overset{\text{WEIGHTED AM-GM}}{\leq}$$

$$\leq\left(\frac{ab\sin C+bc\sin A+ca\sin B}{\sin A+\sin B+\sin C}\right)^{\sin A+\sin B+\sin C}=$$

$$=\left(\frac{2F+2F+2F}{\frac{s}{R}}\right)^{\frac{s}{R}}\overset{\text{MITRINOVIC}}{\leq}\left(\frac{6F}{\frac{s}{R}}\right)^{\frac{1}{R}\cdot\frac{3\sqrt{3}}{2}R}=\left(\frac{6FR}{s}\right)^{\frac{3\sqrt{3}}{2}}=\left(\frac{6rsR}{s}\right)^{\frac{3\sqrt{3}}{2}}=$$

$$=(6rR)^{\frac{3\sqrt{3}}{2}}\leq\left(6\cdot\frac{R}{2}\cdot R\right)^{\frac{3\sqrt{3}}{2}}=(3R^2)^{\frac{3\sqrt{3}}{2}}$$

Equality holds for $a=b=c$.

34. In ΔABC the following relationship holds:

$$\frac{r_a+2r_b}{h_c}+\frac{r_b+2r_c}{h_a}+\frac{r_c+2r_a}{h_b}\geq 9$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{r_a+2r_b}{h_c}+\frac{r_b+2r_c}{h_a}+\frac{r_c+2r_a}{h_b}=\sum_{cyc}\frac{r_a+2r_b}{h_c}=\sum_{cyc}\frac{r_a}{h_c}+2\sum_{cyc}\frac{r_b}{h_c}\overset{\text{AM-GM}}{\geq}$$

$$\geq 3\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}+2\cdot 3\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}=9\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}\geq 9\Leftrightarrow\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}\geq 1\Leftrightarrow r_ar_br_c\geq h_ah_bh_c$$

$$\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}\geq\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c},\qquad abc\geq 8(s-a)(s-b)(s-c)$$

$$abc\cdot s\geq 8s(s-a)(s-b)(s-c),\qquad 4RF\cdot s\geq 8F^2$$

$$Rs\geq 2F, Rs\geq 2rs, R\geq 2r\ \text{(Euler)}$$

Equality holds for $a=b=c$.

35. In $a,b,c>0; abc=1$ then:

$$(a^3+1)^a(b^3+1)^b(c^3+1)^c\geq 8$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$(a^3+1)^a(b^3+1)^b(c^3+1)^c\geq 8$$

$$\ln[(a^3+1)^a(b^3+1)^b(c^3+1)^c]\geq\ln 8$$

$$a\ln(a^3+1)+b\ln(b^3+1)+c\ln(c^3+1)\geq 3\ln 2\ \text{(to prove)}$$

Let be $f:(0,\infty)\to\mathbb{R}; f(x)=x\ln(1+x^3)$

$$f'(x)=\ln(1+x^3)+x\cdot\frac{3x^2}{1+x^3}=\ln(1+x^3)+\frac{3x^3}{1+x^3}>$$

$$f'(x)>0\Rightarrow f\text{ increasing}$$

$$f''(x)=\frac{3x^2}{1+x^3}+\frac{9x^2(1+x^3)-3x^3\cdot 3x^2}{(1+x^3)^2}=\frac{3x^2}{1+x^3}+\frac{9x^2}{(1+x^3)^2}>0$$

$$f''(x)>0\Rightarrow f\text{ convex}$$

$$a\ln(a^3+1)+b\ln(b^3+1)+c\ln(c^3+1)=$$

$$= f(a)+f(b)+f(c) \overset{\text{JENSEN}}{\geq} 3f\left(\frac{a+b+c}{3}\right) \overset{\text{AM-GM}}{\geq} 3f(\sqrt[3]{abc}) = 3f(\sqrt[3]{1}) = 3f(1) = 3\ln 2$$

Equality holds for $a = b = c = 1$.

36. In ΔABC the following relationship holds:

$$\frac{(a+b)^2}{h_c} + \frac{(b+c)^2}{h_a} + \frac{(c+a)^2}{h_b} \geq 48r$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{(a+b)^2}{h_c} + \frac{(b+c)^2}{h_c} + \frac{(c+a)^2}{h_b} = \sum_{cyc} \frac{(a+b)^2}{h_c} \geq$$

$$\overset{\text{AM-GM}}{\geq} 3\sqrt[3]{\frac{((a+b)(b+c)(c+a))^2}{h_a h_b h_c}} \overset{\text{CESARO}}{\geq} 3\sqrt[3]{\frac{(8abc)^2}{h_a h_b h_c}} = 3\sqrt[3]{\frac{64 \cdot (4RF)^2}{\frac{2F}{a} \cdot \frac{2F}{b} \cdot \frac{2F}{c}}} =$$

$$= \frac{3}{2F} \cdot 4\sqrt[3]{(4RF)^2 \cdot abc} = \frac{6}{F}\sqrt[3]{(4RF)^3} = \frac{6}{F} \cdot 4RF = 24R \overset{\text{EULER}}{\geq} 24 \cdot 2r = 47r$$

Equality holds for $a = b = c$.

37. In ΔABC the following relationship holds:

$$\frac{(r_a+r_b)^2}{h_c} + \frac{(r_b+r_c)^2}{h_a} + \frac{(r_c+r_a)^2}{h_b} \geq 36r$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{(r_a+r_b)^2}{h_c} + \frac{(r_b+r_c)^2}{h_a} + \frac{(r_c+r_a)^2}{h_b} = \sum_{cyc} \frac{(r_a+r_b)^2}{h_c} \geq$$

$$\overset{\text{AM-GM}}{\geq} 3\sqrt[3]{\frac{((r_a+r_b)(r_b+r_c)(r_c+r_a))^2}{h_a h_b h_c}} \overset{\text{CESARO}}{\geq} 3\sqrt[3]{\frac{(8r_a r_b r_c)^2}{h_a h_b h_c}}$$

$$= 3\sqrt[3]{\frac{64 \cdot (\frac{F}{s-a} \cdot \frac{F}{s-b} \cdot \frac{F}{s-c})^2}{\frac{2F}{a} \cdot \frac{2F}{b} \cdot \frac{2F}{c}}} =$$

$$= 12\sqrt[3]{\frac{F^6}{((s-a)(s-b)(s-c))^2} \cdot \frac{abc}{8F^3}} = \frac{12}{2}\sqrt[3]{\frac{F^3 \cdot 4RF \cdot s^2}{(s(s-a)(s-b)(s-c))^2}} = 6\sqrt[3]{\frac{4Rs^2F^4}{F^4}} =$$

$$= 6\sqrt[3]{Rs^2} \overset{\text{MITRINOVIC}}{\geq} 6\sqrt[3]{4R \cdot 27r^2} \overset{\text{EULER}}{\geq} 6 \cdot 3\sqrt[3]{8r \cdot r^2} = 36r$$

Equality holds for $a = b = c$.

38. In ΔABC the following relationship holds:

$$\frac{(a+b)^2}{r_c} + \frac{(b+c)^2}{r_a} + \frac{(c+a)^2}{r_b} \geq 48r$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{(a+b)^2}{r_c}+\frac{(b+c)^2}{r_a}+\frac{(c+a)^2}{r_b}=\sum_{cyc}\frac{(a+b)^2}{r_c}\overset{\text{AM-GM}}{\geq}$$

$$\geq 3\sqrt[3]{\frac{((a+b)(b+c)(c+a))^2}{r_ar_br_c}}\overset{\text{CESARO}}{\geq}$$

$$\geq 3\sqrt[3]{\frac{(8abc)^2}{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}}=3\sqrt[3]{\frac{64(abc)^2}{\frac{F^3s}{F^2}}}=12\sqrt[3]{\frac{(4RF)^2}{Fs}}=12\sqrt[3]{\frac{16R^2F^2}{Fs}}=$$

$$=12\sqrt[3]{\frac{16R^2F}{s}}=12\sqrt[3]{\frac{16R^2rs}{s}}=12\sqrt[3]{16R^2r}\overset{\text{EULER}}{\geq}12\sqrt[3]{16\cdot 4r^2\cdot r}=$$

$$=12\sqrt[3]{64r^3}=12\cdot 4r=48r$$

Equality holds for $a=b=c$.

39. In ΔABC the following relationship holds:

$$\frac{r_a+r_b}{h_c}+\frac{r_b+r_c}{h_a}+\frac{r_c+r_a}{h_b}\leq\frac{3R}{r}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\sum_{cyc}\frac{r_a+r_b}{h_c}=\sum_{cyc}\frac{\frac{F}{s-a}+\frac{F}{s-b}}{\frac{2F}{c}}=\sum_{cyc}\frac{\frac{1}{s-a}+\frac{1}{s-b}}{\frac{2}{c}}=$$

$$=\frac{1}{2}\sum_{cyc}\frac{c(s-a+s-b)}{(s-a)(s-b)}=\frac{1}{2}\sum_{cyc}\frac{c(a+b+c-a-b)}{(s-a)(s-b)}=$$

$$=\frac{1}{2}\sum_{cyc}\frac{c^2}{(s-a)(s-b)}=\frac{1}{2(s-a)(s-b)(s-c)}\sum_{cyc}c^2(s-c)=$$

$$=\frac{s}{2s(s-a)(s-b)(s-c)}\cdot 4rs(R+r)=\frac{4rs^2(R+r)}{2F^2}=\frac{2rs^2(R+r)}{r^2s^2}=\frac{2(R+r)}{r}\leq$$

$$\leq\frac{2(R+\frac{R}{2})}{r}=\frac{2\cdot\frac{3R}{2}}{r}=\frac{3R}{r}$$

Equality holds for $a=b=c$.

40. In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2}{c}+\frac{m_b^2+m_c^2}{a}+\frac{m_c^2+m_a^2}{b}\geq 9\sqrt{3}r$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{m_a^2+m_b^2}{c}+\frac{m_b^2+m_c^2}{a}+\frac{m_c^2+m_a^2}{b}=$$

$$=\sum_{cyc}\frac{m_a^2+m_b^2}{c}=\sum_{cyc}\frac{2(b^2+c^2)-a^2+2(a^2+c^2)-b^2}{4c}=$$

$$=\sum_{cyc}\frac{b^2+a^2+4c^2}{4c}=\frac{1}{4}\sum_{cyc}\frac{b^2}{c}+\frac{1}{4}\sum_{cyc}\frac{a^2}{c}+\sum_{cyc}c\geq$$
$$\overset{\text{BERGSTROM}}{\geq}\frac{1}{4}\cdot\frac{(a+b+c)^2}{a+b+c}+\frac{1}{4}\cdot\frac{(a+b+c)^2}{a+b+c}+2s=$$
$$=\frac{a+b+c}{4}+\frac{a+b+c}{4}+2s=\frac{2s}{4}+\frac{2s}{4}+2s=\frac{4s}{4}+2s=3s\overset{\text{MITRINOVIC}}{\geq}$$
$$\geq 3\cdot 3\sqrt{3}r=9\sqrt{3}r$$

Equality holds for $a=b=c$.

41. In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2}{c^2}+\frac{m_b^2+m_c^2}{a^2}+\frac{m_c^2+m_a^2}{b^2}\geq\frac{9}{2}$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{m_a^2+m_b^2}{c^2}+\frac{m_b^2+m_c^2}{a^2}+\frac{m_c^2+m_a^2}{b^2}=$$
$$=\sum_{cyc}\frac{m_a^2+m_b^2}{c^2}=\sum_{cyc}\frac{2(b^2+c^2)-a^2+2(a^2+c^2)-b^2}{4c^2}=$$
$$=\sum_{cyc}\frac{b^2+a^2+4c^2}{4c^2}=\frac{1}{4}\sum_{cyc}\frac{b^2}{c^2}+\frac{1}{4}\sum_{cyc}\frac{a^2}{c^2}+\sum_{cyc}\frac{4c^2}{4c^2}\geq$$
$$\overset{\text{AM-GM}}{\geq}\frac{1}{4}\cdot 3\sqrt[3]{\frac{b^2}{c^2}\cdot\frac{c^2}{a^2}\cdot\frac{a^2}{b^2}}+\frac{1}{4}\cdot 3\sqrt[3]{\frac{a^2}{c^2}\cdot\frac{b^2}{a^2}\cdot\frac{c^2}{b^2}}+3=\frac{3}{4}+\frac{3}{4}+3=\frac{6}{4}+3=\frac{3}{2}+3=\frac{9}{2}$$

Equality holds for $a=b=c$.

42. In ΔABC the following relationship holds:

$$\frac{r_a^2+r_b^2}{c}+\frac{r_b^2+r_c^2}{a}+\frac{r_c^2+r_a^2}{b}\geq 9\sqrt{3}r$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{r_a^2+r_b^2}{c}+\frac{r_b^2+r_c^2}{a}+\frac{r_c^2+r_a^2}{b}=\sum_{cyc}\frac{r_a^2+r_b^2}{c}=\sum_{cyc}\frac{r_a^2}{c}+\sum_{cyc}\frac{r_b^2}{c}\overset{\text{BERGSTROM}}{\geq}$$
$$\geq\frac{(r_a+r_b+r_c)^2}{a+b+c}+\frac{(r_a+r_b+r_c)^2}{a+b+c}=\frac{2(r_a+r_b+r_c)^2}{a+b+c}=$$
$$=\frac{2(4R+r)^2}{2s}=\frac{(4R+r)^2}{s}\overset{\text{DOUCET}}{\geq}\frac{(4R+r)^2}{\frac{4R+r}{\sqrt{3}}}=\sqrt{3}(4R+r)\overset{\text{EULER}}{\geq}\sqrt{3}(4\cdot 2r+r)=9\sqrt{3}r$$

Equality holds for $a=b=c$.

43. In ΔABC the following relationship holds:

$$\frac{a^2+b^2}{r_c}+\frac{b^2+c^2}{r_a}+\frac{c^2+a^2}{r_b}\geq 24r$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$\frac{a^2+b^2}{r_c}+\frac{b^2+c^2}{r_a}+\frac{c^2+a^2}{r_b}=\sum_{cyc}\frac{a^2+b^2}{r_c}\overset{\text{AM-GM}}{\geq}\sum_{cyc}\frac{2ab}{r_c}=2\sum_{cyc}\frac{ab}{r_c}\overset{\text{AM-GM}}{\geq}$$

$$2\cdot 3\sqrt[3]{\frac{(abc)^2}{r_ar_br_c}}=6\sqrt[3]{\frac{16r^2F^2}{Fs}}=6\cdot 2\sqrt[3]{\frac{2R^2F}{s}}=12\sqrt[3]{2R^2r}\geq$$

$$\geq 12\sqrt[3]{2\cdot 4r^2\cdot r}=12\cdot 2r=24r$$

Equality holds for $a=b=c$.

44. In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{r_c}+\frac{b^4+c^4}{r_a}+\frac{c^4+a^4}{r_b}\geq 288r^3$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru - Romania

$$r_ar_br_c=\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}=\frac{F^3s}{s(s-a)(s-b)(s-c)}=\frac{F^3s}{F^2}=Fs$$

$$\frac{a^4+b^4}{r_c}+\frac{b^4+c^4}{r_a}+\frac{c^4+a^4}{r_b}=\sum_{cyc}\frac{a^4+b^4}{r_c}\overset{\text{AM-GM}}{\geq}2\sum_{cyc}\frac{a^2b^2}{r_c}\overset{\text{AM-GM}}{\geq}2\cdot 3\cdot\sqrt[3]{\frac{(abc)^4}{r_ar_br_c}}=$$

$$=6abc\cdot\sqrt[3]{\frac{abc}{Fs}}=6\cdot 4RF\cdot\sqrt[3]{\frac{4RF}{Fs}}=24RF\sqrt[3]{\frac{4R}{s}}\overset{\text{MITRINOVIC}}{\geq}24RF\cdot\sqrt[3]{\frac{4}{s}\cdot\frac{2}{3\sqrt{3}}s}=$$

$$=24Rrs\cdot\frac{2}{\sqrt{3}}\overset{\text{MITRINOVIC}}{\geq}24R\cdot 3\sqrt{3}r\cdot\frac{2}{\sqrt{3}}=24\cdot 6Rr\overset{\text{EULER}}{\geq}144\cdot 2r\cdot r=288r^3$$

Equality holds for $a=b=c$.

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