# RMM - Cyclic Inequalities Marathon 2101 - 2200

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**2101.**

$$If \begin{cases} a_1,a_2,\dots,a_n\in R^+ \\ b_1,b_2,\dots,b_n\in R^+ \end{cases} and\ p,q>0\ \frac{1}{p}+\frac{1}{q}=1\ then\ prove$$

$$\left[\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}a_i\right)^2+\frac{n^2}{4}\left(\sum_{i=1}^{n}b_i^q\right)^{\frac{1}{q}}\right]\sum_{i=1}^{n}\frac{1}{a_i}\ge n^3\sqrt{\sum_{i=1}^{n}a_ib_i}$$

*Proposed by Khaled Abd Imouti-Syria*

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$$LHS=\left[\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}a_i\right)^2+\frac{n^2}{4}\left(\sum_{i=1}^{n}b_i^q\right)^{\frac{1}{q}}\right]\sum_{i=1}^{n}\frac{1}{a_i}=[A+B]\sum_{i=1}^{n}\frac{1}{a_i}$$

$$A+B\overset{AM-GM}{\ge}2\sqrt{AB}\rightarrow A+B\ge 2\sqrt{\left[\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}a_i\right)^2\frac{n^2}{4}\left(\sum_{i=1}^{n}b_i^q\right)^{\frac{1}{q}}\right]}=$$

$$=n\sqrt{\left[\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}a_i\right)^2\left(\sum_{i=1}^{n}b_i^q\right)^{\frac{1}{q}}\right]}=n\sum_{i=1}^{n}a_i\sqrt{\left[\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}b_i^q\right)^{\frac{1}{q}}\right]}$$

$$0<p,q,\quad \frac{1}{p}+\frac{1}{q}=1,\quad \left[\left(\sum_{i=1}^{n}a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}b_i^q\right)^{\frac{1}{q}}\right]\overset{Holders}{\ge}\sum_{i=1}^{n}a_ib_i$$

$$A+B\ge n\sum_{i=1}^{n}a_i\sqrt{\left[\sum_{i=1}^{n}a_ib_i\right]}\rightarrow LHS\ge[A+B]\sum_{i=1}^{n}\frac{1}{a_i}$$

$$LHS\ge n\sum_{i=1}^{n}a_i\sum_{i=1}^{n}\frac{1}{a_i}\sqrt{\left[\sum_{i=1}^{n}a_ib_i\right]}\rightarrow\sum_{i=1}^{n}a_i\sum_{n=1}^{n}\frac{1}{a_i}\overset{Bunyakovski}{\ge}n^2$$

$$LHS\ge n^3\sqrt{\sum_{i=1}^{n}a_ib_i}\ \ Equality\ holds\ for\begin{cases}a_1=a_2=\dots=a_n\\ b_1=b_2=\dots=b_n\end{cases}$$

$$\left[\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} a_i\right)^2+\frac{n^2}{4}\left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}\right]\sum_{i=1}^{n}\frac{1}{a_i}\ge n^3\sqrt{\sum_{i=1}^{n} a_i b_i}\quad (Q.E.D)$$

**2102.** If $a, b, c > 0, a+b+c=3$ and $\lambda \ge 0$ then :

$$\sum_{cyc}\frac{b^2+c^2}{\lambda+a^2}\ge\frac{6}{\lambda+1}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{For all } x>0, \lambda\ge 0, \frac{1}{x+\lambda}-\frac{1}{\lambda+1}+\frac{x-1}{(\lambda+1)^2}=\frac{1-x}{(\lambda+1)(x+\lambda)}-\frac{1-x}{(\lambda+1)^2}$$

$$=\frac{1-x}{\lambda+1}\cdot\left(\frac{1}{x+\lambda}-\frac{1}{\lambda+1}\right)=\frac{(1-x)^2}{(\lambda+1)^2(x+\lambda)}\ge 0 \therefore \frac{1}{x+\lambda}\ge\frac{1}{\lambda+1}-\frac{x-1}{(\lambda+1)^2} \text{ and}$$

$$\text{with } x\equiv a^2, \frac{1}{a^2+\lambda}\ge\frac{1}{\lambda+1}-\frac{a^2-1}{(\lambda+1)^2}$$

$$\Rightarrow\frac{b^2+c^2}{\lambda+a^2}\ge\frac{(b^2+c^2)(\lambda+1)}{(\lambda+1)^2}-\frac{(a^2-1)(b^2+c^2)}{(\lambda+1)^2}\text{ and analogs}$$

$$\Rightarrow\sum_{cyc}\frac{b^2+c^2}{\lambda+a^2}\ge\sum_{cyc}\frac{(b^2+c^2)(\lambda+1)-(a^2-1)(b^2+c^2)}{(\lambda+1)^2}\overset{?}{\ge}\frac{6(\lambda+1)}{(\lambda+1)^2}$$

$$\Leftrightarrow\lambda\left(\sum_{cyc}a^2-3\right)+2\sum_{cyc}a^2-\sum_{cyc}a^2b^2-3\underset{(*)}{\underbrace{\overset{?}{\ge}}}0 \text{ and, now, } 2\sum_{cyc}a^2-\sum_{cyc}a^2b^2-3$$

$$=\frac{2}{9}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)^2-\sum_{cyc}a^2b^2-\frac{3}{81}\left(\sum_{cyc}a\right)^4\left(\because\sum_{cyc}a=3\right)$$

$$=\frac{1}{27}\left(5\sum_{cyc}a^4+8\sum_{cyc}a^3b+8\sum_{cyc}ab^3-21\sum_{cyc}a^2b^2\right)\ge 0;\text{since}$$

$$5\sum_{cyc}a^4\ge 5\sum_{cyc}a^2b^2\text{ and }8\sum_{cyc}a^3b+8\sum_{cyc}ab^3\overset{AM-GM}{\ge}16\sum_{cyc}a^2b^2$$

$$\therefore 2\sum_{cyc} a^2 - \sum_{cyc} a^2b^2 - 3 \overset{①}{\geq} 0 \text{ and } \sum_{cyc} a^2 - 3 \geq \frac{1}{3}\left(\sum_{cyc} a\right)^2 - 3 = \frac{9}{3} - 3 = 0$$

$$\text{and } \because \lambda \geq 0 \therefore \lambda\left(\sum_{cyc} a^2 - 3\right) \overset{②}{\geq} 0 \text{ and so, } ① + ② \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{b^2 + c^2}{\lambda + a^2} \geq \frac{6}{\lambda + 1}\ \forall a, b, c > 0 \,|\, a + b + c = 3 \text{ and } \lambda \geq 0,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

**2103.** If $a, b, c > 0, abc = 1$ and $0 \leq \lambda \leq \frac{5}{3}$ then :

$$\sum_{cyc}\left(a + \frac{\lambda}{b}\right)^2 \geq \frac{3}{4}(\lambda + 1)^2(a + b + c + 1)$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc}\left(a + \frac{\lambda}{b}\right)^2 = \sum_{cyc} a^2 + \lambda^2\sum_{cyc}\frac{1}{a^2} + 2\lambda\sum_{cyc}\frac{a}{b} \geq$$

$$\sum_{cyc} a^2 + \frac{\lambda^2}{a^2b^2c^2}\sum_{cyc} a^2b^2 + \frac{2\lambda}{\sqrt[3]{abc}}.\sum_{cyc} a \geq$$

$$\frac{1}{3}\left(\sum_{cyc} a\right)^2 + \lambda^2 abc\sum_{cyc} a + 2\lambda\left(\sum_{cyc} a\right) (\because abc = 1)$$

$$= \frac{1}{3}\left(\sum_{cyc} a\right)^2 + \lambda^2\sum_{cyc} a + 2\lambda\left(\sum_{cyc} a\right) (\because abc = 1) \overset{?}{\geq} \frac{3}{4}(\lambda + 1)^2\left(\sum_{cyc} a + 1\right)$$

$$\Leftrightarrow 4\left(\sum_{cyc} a\right)^2 + 12\lambda^2\sum_{cyc} a + 24\lambda\left(\sum_{cyc} a\right) \overset{?}{\geq} (9\lambda^2 + 18\lambda + 9)\left(\sum_{cyc} a + 1\right)$$

$$\Leftrightarrow \left(\left(\sum_{cyc} a\right)^2 - 9\right) + 3\left(\sum_{cyc} a\right)^2 + 12\lambda^2\sum_{cyc} a + 24\lambda\left(\sum_{cyc} a\right) \overset{?}{\underset{(*)}{\geq}}$$

$$9\lambda^2+18\lambda+(9\lambda^2+18\lambda+9)\left(\sum_{cyc} a\right)$$

Now, since $\sum_{cyc} a \overset{AM-GM}{\geq} 3.\sqrt[3]{abc} = 3 \Rightarrow \left(\sum_{cyc} a\right)^2 - 9 \geq 0$ and hence,

$3\left(\sum_{cyc} a\right)^2 \geq 9\sum_{cyc} a \therefore$ in order to prove $(*)$, it suffices to prove :

$$9\sum_{cyc} a + 12\lambda^2\sum_{cyc} a + 24\lambda\left(\sum_{cyc} a\right) \overset{?}{\geq} 9\lambda^2+18\lambda+(9\lambda^2+18\lambda+9)\left(\sum_{cyc} a\right)$$

$\Leftrightarrow (3\lambda^2+6\lambda)\left(\sum_{cyc} a\right) \overset{?}{\geq} 9\lambda^2+18\lambda \to$ true $\because \sum_{cyc} a \geq 3$ (shown before) and

$\because \lambda \geq 0 \Rightarrow (*)$ is true $\therefore \sum_{cyc}\left(a+\frac{\lambda}{b}\right)^2 \geq \frac{3}{4}(\lambda+1)^2(a+b+c+1)$

$\forall\, a,b,c>0 \,|\, abc=1$ and $0 \leq \lambda \leq \frac{5}{3}$, $''=''$ iff $a=b=c=1$ (QED)

**2104.** If $a,b,c>0$, $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $\lambda \geq 2$ then :

$$\sum_{cyc}\frac{1}{1+\lambda bc} \geq \frac{3}{\lambda+1}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc}\frac{1}{1+\lambda bc} = \sum_{cyc}\frac{a^2}{a^2+\lambda a^2bc} \overset{Bergstrom}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2 + \lambda abc\sum_{cyc} a}$$

$$= \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a^2 + \lambda\sum_{cyc} ab}\left(\because \sum_{cyc} a = \sum_{cyc}\frac{1}{a}\right) \overset{?}{\geq} \frac{3}{\lambda+1}$$

$$\Leftrightarrow \lambda\left(\sum_{cyc} a\right)^2 + \left(\sum_{cyc} a\right)^2 \overset{?}{\geq} 3\sum_{cyc} a^2 + 3\lambda\sum_{cyc} ab$$

$$\Leftrightarrow \lambda\left(\sum_{cyc} a^2-\sum_{cyc} ab\right) \overset{?}{\geq} 2\left(\sum_{cyc} a^2-\sum_{cyc} ab\right) \Leftrightarrow \frac{(\lambda-2)}{2}.\left(\sum_{cyc}(a-b)^2\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because \lambda \geq 2 \therefore \sum_{cyc}\frac{1}{1+\lambda bc} \geq \frac{3}{\lambda+1}\ \forall\, a,b,c>0 \,\Big|\, a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

$$\text{and } \lambda \geq 2, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

**2105.** **If $x,y,z>0, \sum_{cyc}\frac{\lambda+1}{\lambda^2+x^2}=3$ and $\lambda \geq 1$ then :**

$$\sum_{cyc}\frac{\lambda-x}{\lambda^2-\lambda x+x^2} \geq 0$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{1}{\lambda}.\sum_{cyc}\frac{(\lambda^2-\lambda x+x^2)-x^2}{\lambda^2-\lambda x+x^2}=\frac{1}{\lambda}.\left(3-\sum_{cyc}\frac{x^2}{\lambda^2-\lambda x+x^2}\right)$$

$$\overset{AM-GM}{\geq}\frac{1}{\lambda}.\left(3-\sum_{cyc}\frac{x^2}{\lambda^2-\frac{\lambda^2+x^2}{2}+x^2}\right)=\frac{1}{\lambda}.\left(3-\sum_{cyc}\frac{2(x^2+\lambda^2-\lambda^2)}{\lambda^2+x^2}\right)$$

$$=\frac{1}{\lambda}.\left(-3+2\lambda^2.\sum_{cyc}\frac{1}{\lambda^2+x^2}\right)=\frac{1}{\lambda}.\left(-3+2\lambda^2.\frac{3}{\lambda+1}\right)\left(\because \sum_{cyc}\frac{\lambda+1}{\lambda^2+x^2}=3\right)$$

$$=\frac{3}{\lambda(\lambda+1)}.(2\lambda^2-\lambda-1)=\frac{3(\lambda-1)(2\lambda+1)}{\lambda(\lambda+1)} \geq 0\ (\because \lambda \geq 1)$$

$$\therefore \sum_{cyc}\frac{\lambda-x}{\lambda^2-\lambda x+x^2} \geq 0\ \forall\, x,y,z>0 \,\Big|\, \sum_{cyc}\frac{\lambda+1}{\lambda^2+x^2}=3 \text{ and } \lambda \geq 1,$$

$$''='' \text{ iff } x=y=z=\lambda=1 \text{ (QED)}$$

**2106.** **If $a,b>0$ then:**

$$\frac{a^6+b^6}{a^3\,b^3}+\frac{2ab}{a^2+b^2} \geq 3$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Tapas Das-India*

$$Let\ x=\frac{a^2+b^2}{ab} \overset{AM-GM}{\geq} \frac{2ab}{ab}=2\ (1)$$

$$\frac{a^6+b^6}{a^3\,b^3}=\frac{(a^2+b^2)^3-3a^2b^2(a^2+b^2)}{a^3b^3}=\left(\frac{a^2+b^2}{ab}\right)^3-3\frac{a^2+b^2}{ab}=x^3-3x$$

$$We\ need\ to\ show:$$

$$\frac{a^6+b^6}{a^3\,b^3}+\frac{2ab}{a^2+b^2}\geq 3$$

$$or, x^3-3x+\frac{2}{x}\geq 3\ or, x^4-3x^2-3x+2\geq 0\ or, (x-2)(x^3+2x^2+x-1)\geq 0$$

$$true\ as\ x\geq 2$$

**Equality holds for a=b=1.**

**2107.** If $a,b,c>0, abc=1$ **then:**

$$a+b+c+\frac{1}{ab+bc+ca}\geq\frac{10}{3}$$

***Proposed by Nguyen Hung Cuong-Vietnam***

***Solution by Tapas Das-India***

$$abc=1\ then\ a+b+c\overset{AM-GM}{\geq}3\sqrt[3]{abc}=3$$

$$We\ need\ to\ show:$$

$$a+b+c+\frac{1}{ab+bc+ca}\geq\frac{10}{3}$$

$$a+b+c+\frac{1}{\frac{(a+b+c)^2}{3}}\geq\frac{10}{3}$$

$$x+\frac{3}{x^2}\overset{x=a+b+c\geq 3}{\geq}\frac{10}{3}$$

$$3x^3-10x^2+9\geq 0$$

$$(x-3)(3x^2-x-3)\geq 0\ true\ as\ x\geq 3$$

$$Equality\ holds\ for\ a=b=c=1.$$

**2108.** If $a,b,c>0$ **then:**

$$\frac{a^6+b^6+c^6}{a^2b^2c^2}+\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2\geq 4$$

***Proposed by Nguyen Hung Cuong-Vietnam***

*Solution by Tapas Das-India*

$$ab+bc+ca\overset{AM-GM}{\geq}3\sqrt[3]{a^2b^2c^2}\ or,\frac{(ab+bc+ca)^3}{27}\geq a^2b^2c^2\ (1)$$

$$a^6+b^6+c^6=\frac{(a^2)^3}{(1)^2}+\frac{(b^2)^3}{(1)^2}+\frac{(c^2)^3}{(1)^2}\overset{Radon}{\geq}\frac{(a^2+b^2+c^2)^3}{9}\ (2)$$

$$Let\ x=\frac{a^2+b^2+c^2}{ab+bc+ca}\geq 1\ as\ a^2+b^2+c^2\geq\ ab+bc+ca$$

$$We\ need\ to\ show:$$

$$\frac{a^6+b^6+c^6}{a^2b^2c^2}+\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2\geq 4$$

$$\frac{\frac{(a^2+b^2+c^2)^3}{9}}{\frac{(ab+bc+ca)^3}{27}}+\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2\geq 4$$

$$3\left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^3+\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right)^2\geq 4$$

$$3x^3+\frac{1}{x^2}\geq 4\ or,3x^5-4x^2+1\geq 0$$

$$(x-1)(3x^4+3x^3+3x^2-1)\geq 0\ true\ as\ x\geq 1$$

Equality holds for a=b=c

**2109.** If $a,b,c>0, a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $\lambda\geq 2$ then :

$$\sum_{cyc}\frac{1}{a^2+\lambda}\leq\frac{3}{\lambda+1}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc}\frac{1}{a^2+\lambda}=\frac{\sum_{cyc}a^2b^2+2\lambda\sum_{cyc}a^2+3\lambda^2}{a^2b^2c^2+\lambda\sum_{cyc}a^2b^2+\lambda^2\sum_{cyc}a^2+\lambda^3}\overset{?}{\leq}\frac{3}{\lambda+1}$$

$$\Leftrightarrow(2\lambda-1)\sum_{cyc}a^2b^2+(\lambda^2-2\lambda)\sum_{cyc}a^2+3a^2b^2c^2\overset{?}{\geq}3\lambda^2$$

$$\Leftrightarrow\lambda^2\left(\sum_{cyc}a^2-3\right)+2\lambda\left(\sum_{cyc}a^2b^2-\sum_{cyc}a^2\right)\boxed{\overset{?}{\underset{(*)}{\geq}}}\sum_{cyc}a^2b^2-3a^2b^2c^2$$

$$\text{Now}, \sum_{cyc} ab = abc\sum_{cyc} a \le \frac{1}{3}\left(\sum_{cyc} ab\right)^2 \Rightarrow \sum_{cyc} ab \overset{①}{\ge} 3 \Rightarrow \left(\sum_{cyc} a\right)^2 \ge 9$$

$$\Rightarrow \sum_{cyc} a^2 \ge \frac{1}{3}\left(\sum_{cyc} a\right)^2 \ge 3 \text{ and so}, \lambda^2\left(\sum_{cyc} a^2 - 3\right) + 2\lambda\left(\sum_{cyc} a^2b^2 - \sum_{cyc} a^2\right)$$

$$\overset{\lambda\ge 2}{\ge} 2\lambda\left(\sum_{cyc} a^2 - 3 + \sum_{cyc} a^2b^2 - \sum_{cyc} a^2\right) \overset{\lambda\ge 2}{\ge} 4\left(\sum_{cyc} a^2b^2 - 3\right)$$

$$\left(\because \sum_{cyc} a^2b^2 \ge \frac{1}{3}\left(\sum_{cyc} ab\right)^2 \overset{\text{via } ①}{\ge} 3\right) \text{ and so, in order to prove } (*),$$

$$\text{it suffices to prove}: \sum_{cyc} a^2b^2 + a^2b^2c^2 \boxed{\underset{(**)}{\overset{?}{\ge}}} 4$$

$\boxed{\text{Case 1}}$ $abc \le 1$ and then : $\displaystyle\sum_{cyc} a^2b^2 + a^2b^2c^2 \overset{\text{Darij Grinberg}}{\ge}$

$$2abc\sum_{cyc} a - 1 - a^2b^2c^2 \overset{abc\le 1}{\ge} 2\sum_{cyc} ab - 2\left(\because abc\sum_{cyc} a = \sum_{cyc} ab\right) \overset{\text{via } ①}{\ge} 6 - 2$$
$$= 4 \Rightarrow (**) \text{ is true}$$

$\boxed{\text{Case 2}}$ $abc > 1$ and then : $\displaystyle\sum_{cyc} a^2b^2 + a^2b^2c^2 \overset{abc>1}{>} \frac{1}{3}\left(\sum_{cyc} ab\right)^2 + 1 \overset{\text{via } ①}{\ge} 3 + 1$

$$= 4 \Rightarrow (**) \text{ is true} \therefore \text{combining both cases}, (**) \Rightarrow (*) \text{ is true} \therefore \sum_{cyc} \frac{1}{a^2+\lambda} \le \frac{3}{\lambda+1}$$

$$\forall\, a,b,c > 0 \,\Big|\, a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \text{ and } \lambda \ge 2, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

**2110. If $a, b, c > 0$ and $abc = 1$ then prove that :**

$$a^2 + b^2 + c^2 + \frac{3}{a^3+b^3+c^3} \ge 4$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc} a^2 + \frac{3}{\sum_{cyc} a^3} \overset{abc=1}{=} \frac{\sum_{cyc} a^2}{3.\sqrt[3]{a^2b^2c^2}} + \frac{\sum_{cyc} a^2}{3.\sqrt[3]{a^2b^2c^2}} + \frac{\sum_{cyc} a^2}{3.\sqrt[3]{a^2b^2c^2}} + \frac{3abc}{\sum_{cyc} a^3}$$

$$\overset{AM-GM}{\geq} 4.\sqrt[4]{\frac{3abc\left(\sum_{cyc} a^2\right)^3}{27a^2b^2c^2\left(\sum_{cyc} a^3\right)}} \overset{?}{\geq} 4 \Leftrightarrow \left(\sum_{cyc} a^2\right)^3 \underset{(*)}{\overset{?}{\geq}} 9abc\left(\sum_{cyc} a^3\right)$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of $a$ triangle $XYZ$ with semiperimeter, circumradius and inradius $= s,R,r$ (say); then : $\sum_{cyc} a = s, abc = r^2s, \sum_{cyc} a^2 = s^2-8Rr-2r^2, \sum_{cyc} a^3 = s^3-12Rrs,$

$$\text{and then, } (*) \Leftrightarrow (s^2-8Rr-2r^2)^3 \overset{?}{\geq} 9r^2s(s^3-12Rrs)$$

$$\Leftrightarrow s^6-(24Rr+15r^2)s^4+r^2(192R^2+204Rr+12r^2)s^2-8r^3(4R+r) \boxed{\underset{(\bullet)}{\overset{?}{\geq}}} 0$$

and $\because (s^2-16Rr+5r^2)^3 \overset{Gerretsen}{\geq} 0 \therefore$ in order to prove $(*)$, it suffices to prove :

$$\text{LHS of } (*) \overset{?}{\geq} (s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (24R-30r)s^4-r(576R^2-684Rr+63r^2)s^2+$$

$$r^2(3584R^3-4224R^2r+1104Rr^2-133r^3) \boxed{\underset{(**)}{\overset{?}{\geq}}} 0 \text{ and}$$

$\because (24R-30r)(s^2-16Rr+5r^2)^2 \overset{Gerretsen}{\geq} 0 \therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**) \overset{?}{\geq} (24R-30r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (192R^2-516Rr+237r^2)s^2 \boxed{\underset{(***)}{\overset{?}{\geq}}} r(2560R^3-7296R^2r+4296Rr^2-617r^3)$$

$\boxed{\textit{Case 1}}$ $192R^2-516Rr+237r^2 \geq 0$ and then : LHS of $(***) \overset{Gerretsen}{\geq}$ $(192R^2-516Rr+237r^2)(16Rr-5r^2) \overset{?}{\geq}$ RHS of $(***)$

$$\Leftrightarrow 128t^3-480t^2+519t-142 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(128t^2-224t+71) \overset{?}{\geq} 0$$

$\rightarrow$ true $\because t \overset{Euler}{\geq} 2 \Rightarrow (***)$ is true

$\boxed{\textit{Case 2}}$ $192R^2-516Rr+237r^2 < 0$ and then : LHS of $(***) \overset{Gerretsen}{\geq}$ $(192R^2-516Rr+237r^2)(4R^2+4Rr+3r^2) \overset{?}{\geq}$ RHS of $(***)$

$$\Leftrightarrow 192t^4-964t^3+1689t^2-1224t+332 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)\left((t-2)(192t^2-196t+137)+108\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{Euler}{\geq} 2$$

$\Rightarrow (***)$ is true $\therefore$ combining both cases, $(***) \Rightarrow (**) \Rightarrow (*)$ is true $\forall \Delta XYZ_{s,R,r}$

$$\Rightarrow (\bullet) \text{ is true} \therefore a^2+b^2+c^2+\frac{3}{a^3+b^3+c^3} \geq 4 \ \forall a,b,c>0 \mid abc=1,$$

$'' ='' $ iff $a=b=c=1$ (QED)

**2111.** *If* $a,b,c>0, abc=1$ *then prove that*

$$\frac{a-1}{b^2+c^2}+\frac{b-1}{a^2+c^2}+\frac{c-1}{a^2+b^2}\geq 0$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Amin Hajiyev-Azerbaijan*

$$Lemma\ 1:\ Cebyshev\ inequality:\ a_1\geq a_2\geq a_3, b_1\geq b_2\geq b_3$$

$$\frac{a_1b_1+a_2b_2+a_3b_3}{3}\geq\left(\frac{a_1+a_2+a_3}{3}\right)\left(\frac{b_1+b_2+b_3}{3}\right)$$

$$Lemma\ 2:\ a,b,c>0\to\frac{a+b+c}{3}\geq\sqrt[3]{abc}=1, a+b+c\geq 3$$

$$a\geq b\geq c\to\ a-1\geq b-1\geq c-1\ \ \frac{1}{b^2+c^2}\geq\frac{1}{a^2+c^2}\geq\frac{1}{a^2+b^2}$$

$$\frac{a-1}{b^2+c^2}+\frac{b-1}{a^2+c^2}+\frac{c-1}{a^2+b^2}\overset{Cebyshev}{\geq}\frac{1}{3}(a-1+b-1+c-1)\sum_{cyc}\frac{1}{a^2+b^2}$$

$$\sum_{cyc}\frac{a-1}{b^2+c^2}\geq\frac{1}{3}(a+b+c-3)\sum_{cyc}\frac{1}{a^2+b^2}$$

$$a+b+c-3\geq 0\to\sum_{cyc}\frac{a-1}{b^2+c^2}\geq 0\ \ Q.E.D$$

$$Equality\ holds\ for\ \ a=b=c=1$$

**2112.**

*If* $a,b,c>0$ *and* $a^2+b^2+c^2\leq 3$ *then prove*

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{a^2+b^2+c^2}{2}$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Amin Hajiyev-Azerbaijan*

$$Cauchy-Schwarz\ inequality:\ \sum_{i=1}^{n}\frac{x_i^2}{y_i}\geq\frac{(\sum_{i=1}^{n}x_i)^2}{\sum_{i=1}^{n}y_i}$$

$$\begin{cases}x_1=a, x_2=b, x_3=c\\ y_1=a(b+c), y_2=b(a+c), y_3=c(a+b)\end{cases}$$

$$\frac{a^2}{a(b+c)}+\frac{b^2}{b(a+c)}+\frac{c^2}{c(a+b)}\geq\frac{(a+b+c)^2}{2(ab+ac+bc)}$$

$$LHS\geq\frac{a^2+b^2+c^2+2(ab+ac+bc)}{2(ab+ac+bc)}$$

$$LHS \ge \frac{a^2+b^2+c^2}{2(ab+ac+bc)}+1$$
$$(a-b)^2+(b-c)^2+(a-c)^2 \ge 0$$
$$a^2+b^2+c^2 \ge ab+ac+bc$$
$$LHS \ge \frac{1}{2}+1=\frac{3}{2}$$
$$a^2+b^2+c^2 \le 3 \to RHS=\frac{a^2+b^2+c^2}{2} \le \frac{3}{2}$$
$$LHS \ge \frac{3}{2} \ge RHS \to LHS \ge RHS$$
$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \ge \frac{a^2+b^2+c^2}{2} \quad Q.E.D$$

$$\text{Equality holds for } a=b=c=1$$

**2113.**

$$\text{If } x,y,z>0 \text{ and } \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=1 \text{ then prove}$$
$$\frac{x^2y^2}{z(x^2+y^2)}+\frac{y^2z^2}{x(y^2+z^2)}+\frac{x^2z^2}{y(x^2+z^2)} \ge \frac{3\sqrt{3}}{2}$$

***Proposed by Nguyen Hung Cuong-Vietnam***

***Solution by Amin Hajiyev-Azerbaijan***

$$\text{Substitution } \frac{1}{x}=a, \frac{1}{y}=b, \frac{1}{z}=c \to a^2+b^2+c^2=1 \ a,b,c>0$$
$$LHS=\sum_{cyc}\frac{x^2y^2}{z(x^2+y^2)}=\sum_{cyc}\frac{1}{a^2b^2}\cdot\frac{ca^2b^2}{a^2+b^2}=\sum_{cyc}\frac{c}{a^2+b^2}$$
$$\sum_{cyc}(a^2+b^2)=\sum_{cyc}(1-c^2)$$
$$LHS=\frac{c}{1-c^2}+\frac{a}{1-a^2}+\frac{b}{1-b^2}=\sum_{cyc}\frac{c}{1-c^2}$$
$$a^2=t, b^2=u, c^2=v \to t+u+v=1$$
$$\text{Convex function}: f(k)=\frac{\sqrt{k}}{1-k}, \quad \frac{d^2f(k)}{dk^2}>0 \text{ as } k\in(0;1)$$
$$\frac{f(t)+f(u)+f(v)}{3} \overset{JENSEN}{\ge} f\left(\frac{u+v+t}{3}\right)$$
$$\frac{f(a^2)+f(b^2)+f(c^2)}{3} \ge f\left(\frac{a^2+b^2+c^2}{3}\right)$$

$$LHS \geq 3f\left(\frac{1}{3}\right) = 3 \cdot \frac{\sqrt{\frac{1}{3}}}{1-\frac{1}{3}} = \frac{9}{2\sqrt{3}} = \frac{3\sqrt{3}}{2}$$

$$\frac{x^2y^2}{z(x^2+y^2)} + \frac{y^2z^2}{x(y^2+z^2)} + \frac{x^2z^2}{y(x^2+z^2)} \geq \frac{3\sqrt{3}}{2} \quad Q.E.D$$

$$\text{Equality holds for } \frac{1}{x} = \frac{1}{y} = \frac{1}{z} = \frac{1}{\sqrt{3}}$$

**2114.**

**If $a, b, c > 0$ *then prove that*:**

$$\frac{a^3-b^3}{b^2+c^2} + \frac{b^3-c^3}{c^2+a^2} + \frac{c^3-a^3}{a^2+b^2} \geq 0$$

***Proposed by Nguyen Hung Cuong-Vietnam***

***Solution by Soumava Chakraborty-Kolkata-India***

$$\frac{a^3-b^3}{b^2+c^2} + \frac{b^3-c^3}{c^2+a^2} + \frac{c^3-a^3}{a^2+b^2} =$$

$$= \frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \cdot \sum_{cyc}\left((b^3-c^3)\left(b^4 + \sum_{cyc} a^2b^2\right)\right) =$$

$$= \frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \cdot \left(\left(\sum_{cyc} a^2b^2\right)\sum_{cyc}(b^3-c^3) + \sum_{cyc} b^7 - \sum_{cyc} b^4c^3\right) =$$

$$= \frac{1}{(a^2+b^2)(b^2+c^2)(c^2+a^2)} \cdot \left(\sum_{cyc} a^7 - \sum_{cyc} b^4c^3\right) = LHS \to (*)$$

$$\text{Now, } a^7+a^7+a^7+a^7+b^7+b^7+b^7 \overset{AM-GM}{\geq} 7.\sqrt[7]{a^{28}b^{21}} \Rightarrow 4a^7+3b^7 \overset{①}{\geq} 7a^4b^3$$

$$\text{Also, } b^7+b^7+b^7+b^7+c^7+c^7+c^7 \overset{AM-GM}{\geq} 7.\sqrt[7]{b^{28}c^{21}} \Rightarrow 4b^7+3c^7 \overset{②}{\geq} 7b^4c^3$$

$$\text{Again, } c^7+c^7+c^7+c^7+a^7+a^7+a^7 \overset{AM-GM}{\geq} 7.\sqrt[7]{c^{28}a^{21}} \Rightarrow 4c^7+3a^7 \overset{③}{\geq} 7c^4a^3$$

$$\therefore ① + ② + ③ \Rightarrow 7(a^7+b^7+c^7) \geq 7(a^4b^3+b^4c^3+c^4a^3)$$

$$\Rightarrow \sum_{cyc} a^7 - \sum_{cyc} b^4c^3 \overset{(**)}{\geq} 0 \text{ and so, } (*) \text{ and } (**) \Rightarrow LHS \geq 0$$

$$\therefore \frac{a^3-b^3}{b^2+c^2} + \frac{b^3-c^3}{c^2+a^2} + \frac{c^3-a^3}{a^2+b^2} \geq 0\ \forall\, a,b,c > 0, ''='' \ a = b = c \text{ (QED)}$$

**2115.** *If* $a,b,c>0$ *and* $a^2+b^2+c^2=3$ *then*:

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac}\ge\frac{3}{2}$$

*Proposed by Mais Hasanov-Azerbaijan*

*Solution by Amin Hajiyev-Azerbaijan*

**Cauchy-Schwarz inequality:**

$$\left(\sum_{i=1}^{n}a_ib_i\right)^2\le\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)\ as\ a_i,b_i\in R$$

$$a_1=\frac{1}{b_1}=\sqrt{1+ab},a_2=\frac{1}{b_2}=\sqrt{1+bc},a_3=\frac{1}{b_3}=\sqrt{1+ac}$$

$$(a_1\cdot b_1+a_2\cdot b_2+a_3\cdot b_3)^2\le(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$$

$$3^2\le(3+ab+bc+ac)\left(\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac}\right)$$

$$\sum_{cyc}\frac{1}{1+ab}\ge\frac{9}{3+ab+ac+bc}$$

$$We\ know\ (a-b)^2+(b-c)^2+(a-c)^2\ge 0$$

$$a^2+b^2+c^2\ge ab+bc+ac\to 3\ge ab+ac+bc$$

$$\sum_{cyc}\frac{1}{1+ab}\ge\frac{9}{6}=\frac{3}{2}$$

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ac}\ge\frac{3}{2}$$

$$Equality\ holds\ for\ a=b=c=1$$

**2116.** If $x,y,z,\lambda>0,xyz=1$ then :

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+(\lambda+1)(x+y+z)\ge 6\lambda+(2-\lambda)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Since } xyz=1,\text{ so we can assign } x=\frac{a}{b},y=\frac{b}{c},z=\frac{c}{a}\ (a,b,c>0)\text{ and}$$

$$\text{then}: \sum_{cyc}\frac{x}{y}+\sum_{cyc}x\overset{?}{\ge}2\sum_{cyc}\frac{1}{x}\Leftrightarrow\sum_{cyc}\frac{ca}{b^2}+\sum_{cyc}\frac{a}{b}\underset{(*)}{\overset{?}{\ge}}2\sum_{cyc}\frac{b}{a}$$

$$\text{Now, }\sum_{cyc}\frac{ca}{b^2}+\sum_{cyc}\frac{a}{b}=\sum_{cyc}\frac{ca}{b^2}+\sum_{cyc}\frac{c}{a}\overset{AM-GM}{\geq}2.\sum_{cyc}\sqrt{\frac{ca}{b^2}.\frac{c}{a}}=2\sum_{cyc}\frac{c}{b}=2\sum_{cyc}\frac{b}{a}$$

$$\Rightarrow (*)\text{ is true }\therefore\sum_{cyc}\frac{x}{y}+\sum_{cyc}x\geq 2\sum_{cyc}\frac{1}{x}\to (1)\text{ and again, }\sum_{cyc}x+\sum_{cyc}\frac{1}{x}\overset{AM-GM}{\geq}6$$

$$\Rightarrow \lambda\left(\sum_{cyc}x+\sum_{cyc}\frac{1}{x}-6\right)\geq 0\ (\because \lambda>0)\Rightarrow \lambda\sum_{cyc}x\geq 6\lambda-\lambda\sum_{cyc}\frac{1}{x}\to (2)$$

$$\therefore (1)+(2)\Rightarrow \frac{x}{y}+\frac{y}{z}+\frac{z}{x}+(\lambda+1)(x+y+z)\geq 6\lambda+(2-\lambda)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$\forall\, x,y,z,\lambda>0\,|\,xyz=1,''='' \text{ iff } x=y=z=1\ (QED)$$

**2117. If $a,b>0, a+b=2$ then:**

$$a(a^2+b)+b(b^2+a)\geq 4$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Daniel Sitaru-Romania*

$$\begin{cases}a+b=S\\ ab=P\end{cases}\Rightarrow S=2$$

$$2=S=a+b\overset{AM-GM}{\geq}2\sqrt{ab}=2\sqrt{P}\Rightarrow P\leq 1\quad (1)$$

$$a(a^2+b)+b(b^2+a)\geq 4$$

$$a^3+b^3+2ab\geq 4,\quad S^3-3SP+2P\geq 4$$

$$8-6P+2P\geq 4,\qquad 4P\leq 4\overset{(1)}{\Leftrightarrow}P\leq 1$$

Equality holds for $a=b$.

**2118. If $a,b,c>0$ then:**

$$\frac{a}{\sqrt{2a^2+b^2+c^2}}+\frac{b}{\sqrt{2b^2+a^2+c^2}}+\frac{c}{\sqrt{2c^2+b^2+a^2}}\leq\frac{3}{2}$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Tapas Das-India*

$$2a^2+b^2+c^2=\left(\frac{a^2}{1}+\frac{b^2}{1}\right)+\left(\frac{a^2}{1}+\frac{c^2}{1}\right)\overset{Radon}{\geq}$$

$$\geq \frac{(a+b)^2}{2}+\frac{(a+c)^2}{2} \overset{AM-GM}{\geq} (a+b)(a+c)$$

$$\frac{a}{\sqrt{2a^2+b^2+c^2}}+\frac{b}{\sqrt{2b^2+a^2+c^2}}+\frac{c}{\sqrt{2c^2+b^2+a^2}}=$$

$$=\sum\frac{a}{\sqrt{2a^2+b^2+c^2}}\leq\sum\frac{a}{\sqrt{(a+b)(a+c)}}=\sum\sqrt{\frac{a}{a+b}\cdot\frac{a}{a+c}} \overset{AM-GM}{\leq}$$

$$\leq\frac{1}{2}\sum\left(\frac{a}{a+b}+\frac{a}{a+c}\right)=\frac{1}{2}\sum\left(\frac{a}{a+b}+\frac{b}{a+b}\right)=\frac{1}{2}\sum\frac{a+b}{a+b}=\frac{3}{2}$$

Equality holds for an equilateral triangle.

**2119. If $a,b,c>0, a+b+c=3$ and $n\in\mathbb{N}$ then :**

$$\sum_{cyc}\left(\frac{a^3}{9-a(3b+3c+2bc)}\right)^n\geq 3$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Firstly}, 9-a(3b+3c+2bc) \overset{a+b+c=3}{=} 9-2abc-3a(3-a)$$

$$=3a^2-9a+9-2abc\geq 3a^2-9a+9-2\left(\because 3=\sum_{cyc}a \overset{AM-GM}{\geq} 3.\sqrt[3]{abc}\right)>0$$

$$\because \Delta_{3a^2-9a+7}=81-84<0 \therefore 9-a(3b+3c+2bc) \text{ and analogs}>0\to (1)$$

$$\text{Assigning } b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a$$
$$>0 \text{ and } z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z \text{ form sides}$$
$$\text{of a triangle XYZ with semiperimeter, circumradius and inradius}=s,R,r\ (say);$$
$$\text{then}: \sum_{cyc}a=s, abc=r^2s, \sum_{cyc}ab=4Rr+r^2, \sum_{cyc}a^2=s^2-8Rr-2r^2,$$
$$\sum_{cyc}a^3=s^3-12Rrs, \sum_{cyc}a^2b^2=r^2((4R+r)^2-2s^2) \text{ and then},$$
$$\sum_{cyc}\left(a^3(9-b(3c+3a+2ca))(9-c(3a+3b+2ab))\right)-$$
$$3\prod_{cyc}(9-a(3b+3c+2bc)) \boxed{=} \sum_{cyc}\left(a^3\left(9-2abc-(9b-3b^2)\right)\begin{pmatrix}9-2abc-\\(9c-3c^2)\end{pmatrix}\right)$$

$$-3\prod_{cyc}\left(9-2abc-(9a-3a^2)\right)\overset{a+b+c=3}{=}\frac{1}{9}\left(\left(\sum_{cyc}a\right)^3-6abc\right)^2\left(\sum_{cyc}a^3\right)-$$

$$\frac{1}{3}\left(\left(\sum_{cyc}a\right)^3-6abc\right)\left(\frac{9\left((\sum_{cyc}ab)(\sum_{cyc}a^2)-abc\sum_{cyc}a\right)(\sum_{cyc}a)^2}{9}-\frac{3\left((\sum_{cyc}a)(\sum_{cyc}a^2b^2)-abc\sum_{cyc}ab\right)(\sum_{cyc}a)}{3}\right)+$$

$$\frac{81abc(\sum_{cyc}a^2)(\sum_{cyc}a)^4}{81}-\frac{27abc\left((\sum_{cyc}a)(\sum_{cyc}ab)-3abc\right)(\sum_{cyc}a)^3}{27}+\frac{9a^2b^2c^2(\sum_{cyc}ab)(\sum_{cyc}a)}{3}-$$

$$3\left(\frac{\left((\sum_{cyc}a)^3-6abc\right)^3-\left((\sum_{cyc}a)^3-6abc\right)^2\left(3(\sum_{cyc}a)^2-3\sum_{cyc}a^2\right)(\sum_{cyc}a)}{27}\right)$$

$$-\left(\frac{(\sum_{cyc}a)^3-6abc}{3}\right)\left(\frac{81(\sum_{cyc}ab)(\sum_{cyc}a)^4-81\left((\sum_{cyc}a)(\sum_{cyc}ab)-3abc\right)(\sum_{cyc}a)^3+(\sum_{cyc}a^2b^2)(\sum_{cyc}a)^2}{27}\right)$$

$$+\frac{81abc(a+b)(b+c)(c+a)}{27}\left(\sum_{cyc}a\right)^3\boxed{=}\frac{(s^3-6r^2s)^2(s^3-12Rrs)}{9}-$$

$$\frac{(s^3-6r^2s)\left(s^2\left((4Rr+r^2)(s^2-8Rr-2r^2)-r^2s^2\right)-s\left(sr^2((4R+r)^2-2s^2)-r^2s(4Rr+r^2)\right)\right)}{3}+r^2s^5(s^2-8Rr-2r^2)-$$

$$r^2s^4(s(4Rr+r^2)-3r^2s)+3r^4s^3(4Rr+r^2)$$

$$-\frac{(s^3-6r^2s)^3-6s(4Rr+r^2)(s^3-6r^2s)^2}{9}$$

$$-(s^3-6r^2s)\left((4Rr+r^2)s^4-s^3(s(4Rr+r^2)-3r^2s)+s^2r^2((4R+r)^2-2s^2)\right)+$$

$$12Rr^3s^5\overset{?}{\geq}0\Leftrightarrow 2s^4-(28Rr+19r^2)s^2+204Rr^3+159r^4\boxed{\underset{(*)}{\overset{?}{\geq}}}0\ and\ \because P=$$

$$(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\ \therefore \text{in order to prove }(*),\text{it suffices to prove}:$$

$$\text{LHS of }(*)\overset{?}{\geq}P\Leftrightarrow(36R-39r)s^2\boxed{\underset{(**)}{\overset{?}{\geq}}}r(512R^2-524Rr-109r^2)$$

$$\text{Again},(36R-39r)s^2 \geq (36R-39r)\left(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\right)$$

$$\left(\begin{array}{c}\because (R-r)(s^2-16Rr+5r^2) \overset{\text{Rouche}}{\geq} (R-r)\left(2R^2-6Rr+4r^2-2(R-2r)\sqrt{R^2-2Rr}\right)\\ =(R-2r)\left(\left(R-r-\sqrt{R^2-2Rr}\right)^2+r^2\right) \geq r^2(R-2r)\ \left(\because R-2r \overset{\text{Euler}}{\geq} 0\right)\end{array}\right)$$

$$\overset{?}{\geq} \text{RHS of } (**) \Leftrightarrow 64t^3-308t^2+473t-226 \overset{?}{\geq} 0\left(t=\frac{R}{r}\right) \Leftrightarrow$$

$$(t-2)\left((t-2)(64t-52)+9\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (*) \Rightarrow (**) \text{ is true } \overset{\text{via } (1)}{\Rightarrow}$$

$$\sum_{\text{cyc}} \frac{a^3}{9-a(3b+3c+2bc)} \geq 3 \Rightarrow \sum_{\text{cyc}}\left(\frac{a^3}{9-a(3b+3c+2bc)}\right)^n \overset{\text{Holder}}{\geq}$$

$$3\left(\frac{\sum_{\text{cyc}}\frac{a^3}{9-a(3b+3c+2bc)}}{3}\right)^n \geq 3\ (\because n \in \mathbb{N}) \therefore \sum_{\text{cyc}}\left(\frac{a^3}{9-a(3b+3c+2bc)}\right)^n \geq 3$$

$$\forall a,b,c>0 \mid a+b+c=3 \text{ and } n \in \mathbb{N}, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

**2120. If $a,b,c>0$, $a+b+c=3$, $n\in N$ then:**

$$\sum\left(\frac{a^3}{a^2+b^2}\right)^n \geq \frac{3}{2^n}$$

***Proposed by Marin Chirciu-Romania***

***Solution by Tapas Das-India***

$$\sum\frac{a^3}{a^2+b^2}=\sum\left(a-\frac{ab^2}{a^2+b^2}\right)=\sum a-\sum\frac{ab^2}{a^2+b^2} \overset{AM-GM}{\geq}$$

$$\geq \sum a-\sum\frac{ab^2}{2ab}=\sum a-\frac{1}{2}\sum a \overset{a+b+c=3}{=} \frac{3}{2}\ \ (1)$$

$$\sum\left(\frac{a^3}{a^2+b^2}\right)^n=\sum\frac{\left(\frac{a^3}{a^2+b^2}\right)^n}{(1)^{n-1}} \overset{Radon}{\geq} \frac{\left(\sum\frac{a^3}{a^2+b^2}\right)^n}{3^{n-1}} \overset{(1)}{\geq} \frac{\left(\frac{3}{2}\right)^n}{3^{n-1}}=\frac{3}{2^n}$$

Equality holds for a=b=c=1

**2121.** If $a,b,c>0, a+b+c=3$ and $n\in\mathbb{N}^*$ then :

$$\sum_{cyc}\frac{a^{n+1}}{b^{n+1}.c}\ge\frac{1}{3^n}\left(\sum_{cyc}ab^2\right)^{n+1}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$3=\sum_{cyc}a\overset{AM-GM}{\ge}3.\sqrt[3]{abc}\Rightarrow abc\le 1\to(1)\text{ and we have : }\sum_{cyc}ab^2+\sum_{cyc}\frac{b}{a}$$

$$\overset{via\,(1)}{\le}\frac{1}{abc}.\sum_{cyc}ab^2+\frac{1}{abc}.\sum_{cyc}a^2b=\sum_{cyc}\frac{b}{c}+\sum_{cyc}\frac{a}{c}=\sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{b}{a}$$

$$\Rightarrow\sum_{cyc}\frac{a}{b}\ge\sum_{cyc}ab^2\to(2)\text{ and now, since }n\in\mathbb{N}^*\therefore\sum_{cyc}\frac{a^{n+1}}{b^{n+1}.c}=$$

$$\sum_{cyc}\frac{\left(\frac{a}{b}\right)^{n+1}}{c}\overset{Holder}{\ge}\frac{\left(\sum_{cyc}\frac{a}{b}\right)^{n+1}}{3^{n-1}.\sum_{cyc}a}\overset{via\,(2)}{\ge}\frac{\left(\sum_{cyc}ab^2\right)^{n+1}}{3^{n-1}.\sum_{cyc}a}\overset{a+b+c=3}{=}\frac{1}{3^n}\left(\sum_{cyc}ab^2\right)^{n+1}$$

$$\therefore\sum_{cyc}\frac{a^{n+1}}{b^{n+1}.c}\ge\frac{1}{3^n}\left(\sum_{cyc}ab^2\right)^{n+1}\quad\forall\, a,b,c>0\,\middle|\,a+b+c=3\text{ and }n\in\mathbb{N}^*,$$

$$''=''\text{ iff }a=b=c=1\text{ (QED)}$$

**2122.** If $a,b,c>0$ and $abc=1$ then prove the inequality :

$$\sum_{cyc}\frac{8}{(a+c)b+a+b+4}\le 3$$

*Proposed by Neculai Stanciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc}\frac{8}{(a+c)b+a+b+4}\overset{abc=1}{=}\sum_{cyc}\frac{8}{ab+\left(\frac{1}{a}+a\right)+b+4}$$

$$\overset{AM-GM}{\le}\sum_{cyc}\frac{8}{ab+b+2+4}\overset{AM-GM}{\le}\sum_{cyc}\frac{8}{2\sqrt{ab^2}+6}\overset{abc=1}{=}\sum_{cyc}\frac{4}{\sqrt{\frac{b}{c}}+3}=\sum_{cyc}\frac{4}{x+3}$$

$$\left(x=\sqrt{\frac{b}{c}},y=\sqrt{\frac{c}{a}},z=\sqrt{\frac{a}{b}}\right)=\sum_{cyc}\frac{4(y+3)(z+3)}{(x+3)(y+3)(z+3)}\overset{?}{\le}3$$

$$\Leftrightarrow 3xyz+5\sum_{cyc}xy+3\sum_{cyc}x\overset{?}{\ge}27\Leftrightarrow 5\sum_{cyc}xy+3\sum_{cyc}x\underbrace{\overset{?}{\ge}}_{(*)}24\ (\because xyz=1)$$

$$\text{Indeed, }5\sum_{cyc}xy+3\sum_{cyc}x\overset{AM-GM}{\ge}15.\sqrt[3]{x^2y^2z^2}+9.\sqrt[3]{xyz}\overset{xyz=1}{=}24\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\frac{8}{(a+c)b+a+b+4}\le 3\ \forall\, a,b,c>0\,\middle|\, abc=1,$$

$$''=''\text{ iff } a=b=c=1$$

**2123.** If $a,b>0, ab=1$ then:

$$\left(\frac{a}{b+1}\right)^2+\left(\frac{b}{a+1}\right)^2\ge\frac{1}{2}$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Tapas Das-India*

$$\frac{a}{b+1}+\frac{b}{a+1}=\frac{a(a+1)+b(b+1)}{(a+1)(b+1)}=\frac{a^2+b^2+a+b}{1+ab+a+b}\overset{\substack{AM-GM\\ab=1}}{\ge}$$

$$\ge\frac{2ab+a+b}{ab+ab+a+b}=1\ (1)$$

$$\left(\frac{a}{b+1}\right)^2+\left(\frac{b}{a+1}\right)^2=\frac{\left(\frac{a}{b+1}\right)^2}{1}+\frac{\left(\frac{b}{a+1}\right)^2}{1}\overset{Radon}{\ge}\frac{\left(\frac{a}{b+1}+\frac{b}{a+1}\right)^2}{2}\overset{(1)}{\ge}\frac{1}{2}$$

Equality holds for a=b=1

**2124.** If $a,b,c,x,y,z>0$ and

$$2(a+b+c)^2+(x+y+z)^2+1\le 2(a+b+c)(x+y+z+1)$$

then:

$$\frac{(x+b+c)^9}{(x^ay^bz^c+a^xb^yc^z)^8}+\frac{(y+z+a)^9}{(x^by^cz^a+a^yb^zc^x)^8}+\frac{1}{(x^cy^az^b+a^zb^xc^y)^8}\ge\frac{19683}{256}$$

*Proposed by Pavlos Trifon-Greece*

*Solution by Tapas Das-India*

$$2(a+b+c)^2+(x+y+z)^2+1\le 2(a+b+c)(x+y+z+1)\ can\ be\ written\ as$$
$$\big(x+y+z-(a+b+c)\big)^2+\big((a+b+c)-1\big)^2\le 0$$
$$since\ L.H.S\ge 0\ ,so\ only\ possibility\ is$$
$$\big((x+y+z)-(a+b+c)\big)^2=0\ and\ \big((a+b+c)-1\big)^2=0$$
$$which\ implies\ that\ x+y+z=a+b+c=1\ (1)$$

$$x^ay^bz^c+a^xb^yc^z+x^by^cz^a+a^yb^zc^x+x^cy^az^b+a^zb^xc^y=$$
$$=\left(x^ay^bz^c+a^xb^yc^z\right)+\left(x^by^cz^a+a^zb^xc^y\right)+\left(a^yb^zc^x+x^cy^az^b\right)$$

$$=\sum\left(x^ay^bz^c+a^xb^yc^z\right)\overset{AM-GM}{\le}\sum\left(\frac{ax+by+cz}{a+b+c}\right)^{a+b+c}+\sum\left(\frac{ax+by+cz}{x+y+z}\right)^{x+y+z}$$
$$\overset{(1)}{\le}\sum(ax+by+cz)+\sum(ax+by+cz)=2\sum(ax+by+cz)\ (2)$$

$$\frac{(x+b+c)^9}{(x^ay^bz^c+a^xb^yc^z)^8}+\frac{(y+z+a)^9}{(x^by^cz^a+a^yb^zc^x)^8}+\frac{1}{(x^cy^az^b+a^zb^xc^y)^8}=$$

$$=\frac{(x+b+c)^9}{(x^ay^bz^c+a^xb^yc^z)^8}+\frac{(y+z+a)^9}{(x^by^cz^a+a^yb^zc^x)^8}+\frac{1^9}{(x^cy^az^b+a^zb^xc^y)^8}\ge$$

$$\overset{Radon}{\ge}\frac{(x+b+c+y+z+a+1)^9}{(x^ay^bz^c+a^xb^yc^z+x^by^cz^a+a^yb^zc^x+x^cy^az^b+a^zb^xc^y)^8}\ge$$
$$\overset{(1)\&(2)}{\ge}\frac{(3)^9}{\left(2\sum(ax+by+cz)\right)^8}=\frac{3^9}{\big(2(a+b+c)(x+y+z)\big)^8}\overset{(1)}{\ge}\frac{3^9}{2^8}=\frac{19683}{256}$$

$$Equality\ holds\ for\ a=b=c=x=y=z=\frac{1}{3}$$

**2125.** **If $a,b,c>0,|\lambda|\le 2$ then:**

$$\sum(b+c)(a^2+\lambda ab+b^2)\ge\frac{3(\lambda+2)}{4}\prod(a+b)$$

*Proposed by Marin Chirciu-Romania*

*Solution by Tapas Das-India*

$$a^2+\lambda ab+b^2=(a+b)^2-2ab+\lambda ab=(a+b)^2+(\lambda-2)ab\ \overset{\substack{AM-GM\\ as\ |\lambda|\le 2}}{\ge}$$

$$\geq (a+b)^2 + (\lambda-2)\frac{(a+b)^2}{4} = \frac{(\lambda+2)}{4}(a+b)^2$$

$$\sum (b+c)(a^2+\lambda ab+b^2) \geq \sum (b+c)\frac{(\lambda+2)}{4}(a+b)^2 \overset{AM-GM}{\geq}$$

$$\geq \frac{3(\lambda+2)}{4}\sqrt[3]{\prod (b+c)(a+b)^2} = \frac{3(\lambda+2)}{4}\prod (a+b)$$

Equality holds for a=b=c.

**2126. If $a, b, c > 0, \sum_{cyc} a^2b^2 = 1$, then :**

$$\sum_{cyc}\sqrt{a^2+\frac{1}{a^2}} \geq \sum_{cyc} a + \frac{1}{abc}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc}\sqrt{a^2+\frac{1}{a^2}} \overset{?}{\geq} \sum_{cyc} a + \frac{1}{abc}$$

$$\Leftrightarrow \sum_{cyc} a^2 + \sum_{cyc}\frac{1}{a^2} + 2\sum_{cyc}\sqrt{\left(a^2+\frac{1}{a^2}\right)\left(b^2+\frac{1}{b^2}\right)} \underset{(*)}{\overset{?}{\geq}}$$

$$\left(\sum_{cyc} a\right)^2 + \frac{1}{a^2b^2c^2} + \frac{2}{abc}\left(\sum_{cyc} a\right)$$

$$\text{Now, LHS of } (*) \overset{\text{Reverse CBS}}{\geq} \sum_{cyc} a^2 + \sum_{cyc}\frac{1}{a^2} + 2\sum_{cyc} ab + 2\sum_{cyc}\frac{1}{ab}$$

$$= \left(\sum_{cyc} a\right)^2 + \frac{1}{a^2b^2c^2}\cdot\left(\sum_{cyc} a^2b^2\right) + \frac{2}{abc}\left(\sum_{cyc} a\right) \overset{\sum_{cyc} a^2b^2 = 1}{=}$$

$$\left(\sum_{cyc} a\right)^2 + \frac{1}{a^2b^2c^2} + \frac{2}{abc}\left(\sum_{cyc} a\right) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \sqrt{a^2 + \frac{1}{a^2}} \geq \sum_{cyc} a + \frac{1}{abc}\ \forall\, a, b, c > 0 \,\Big|\, \sum_{cyc} a^2b^2 = 1,$$

$$'' ='' \ a = b = c = \frac{1}{\sqrt[4]{3}}$$

**2127.** **If $x, y, z > 0$ and $x + y + z = 1$ then prove that :**

$$\sum_{cyc} \frac{az + by}{(a + b)(x + yz)} \geq \sum_{cyc} \frac{x}{x + yz} \text{ for all } a + b > 0$$

*Proposed by Neculai Stanciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc} \frac{az + by}{(a + b)(x + yz)} - \sum_{cyc} \frac{x}{x + yz} \overset{x+y+z=1}{=}$$

$$\sum_{cyc} \frac{az + by}{(a + b)(1 - y - z + yz)} - \sum_{cyc} \frac{x}{1 - y - z + yz}$$

$$= \sum_{cyc} \frac{az + by}{(a + b)(1 - y)(1 - z)} - \sum_{cyc} \frac{x}{(1 - y)(1 - z)}$$

$$\overset{x+y+z=1}{=} \sum_{cyc} \frac{az + by}{(a + b)(z + x)(x + y)} - \sum_{cyc} \frac{x}{(z + x)(x + y)}$$

$$= \frac{1}{(x + y)(y + z)(z + x)} \cdot \left(\sum_{cyc} \frac{(az + by)(y + z)}{a + b} - \sum_{cyc} (x(y + z))\right)$$

$$= \frac{1}{(x + y)(y + z)(z + x)} \cdot \left(\frac{a + b}{a + b} \cdot \left(\sum_{cyc} x^2 + \sum_{cyc} xy\right) - 2\sum_{cyc} xy\right)$$

$$= \frac{1}{(x + y)(y + z)(z + x)} \cdot \left(\sum_{cyc} x^2 - \sum_{cyc} xy\right) (\because a + b \neq 0 \text{ as } a + b > 0) \geq 0$$

$$(\because x, y, z > 0 \Rightarrow (x + y)(y + z)(z + x) > 0) \therefore \sum_{cyc} \frac{az + by}{(a + b)(x + yz)} \geq \sum_{cyc} \frac{x}{x + yz}$$

$$\forall\, x,y,z>0\,\middle|\,x+y+z=1 \text{ and } \forall\, a,b\,\middle|\,a+b\neq 0,''=''\ \text{ iff } x=y=z=\frac{1}{3}\ (\text{QED})$$

**2128.** If $x,y>0, x\neq y$ then:

$$\left(\frac{(x+y)xy}{(x-y)^2}+x+y\right)\left(\frac{x+y}{(x-y)^2}+\frac{1}{x}+\frac{1}{y}\right)\frac{(x+y)^2}{xy}\geq 81$$

*Proposed by Neculai Stanciu-Romania*

*Solution by Tapas Das-India*

$$Let\ a=x+y\,,b=xy\ then\ (x-y)^2=(x+y)^2-4xy=a^2-4b$$

$$a^2=(x+y)^2\overset{AM-GM}{>}4xy\ or\ a^2>4b\ or\ \frac{a^2}{b}>4\ or\ t\overset{t=\frac{a^2}{b}}{=}4\ (1)$$

$$\frac{(x+y)xy}{(x-y)^2}+x+y=\frac{ab}{a^2-4b}+a\ and\ \frac{x+y}{(x-y)^2}+\frac{1}{x}+\frac{1}{y}=\frac{a}{a^2-4b}+\frac{a}{b},\frac{(x+y)^2}{xy}=\frac{a^2}{b}$$

$$L.H.S=\left(\frac{ab}{a^2-4b}+a\right)\left(\frac{a}{a^2-4b}+\frac{a}{b}\right).\frac{a^2}{b}=\frac{a^4}{b^2}.\frac{(a^2-3b)^2}{(a^2-4b)^2}\overset{t=\frac{a^2}{b}}{=}\frac{t^2(t-3)^2}{(t-4)^2}$$

$$We\ need\ to\ show:$$

$$\frac{t^2(t-3)^2}{(t-4)^2}\geq 81\ or,\frac{t(t-3)}{t-4}\geq 9\ or,\frac{t(t-3)}{t-4}-9\geq 0\ or,\frac{(t-6)^2}{t-4}\geq 0$$

$$which\ is\ true\ as\ t>4\big(from\ (1)\big)$$

$$and\ equality\ occurs\ when\ t=6\ or,\frac{a^2}{b}=6\ or,a^2=6b$$

**2129.** If $a,b>0, ab=1$ then:

$$a^b b^a\leq 1$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Daniel Sitaru-Romania*

$$a^b b^a\overset{AM-GM}{\leq}\frac{ab+ba}{a+b}=\frac{1+1}{a+b}=\frac{1}{\frac{a+b}{2}}\overset{AM-GM}{\leq}\frac{1}{\sqrt{ab}}=1$$

Equality holds for $a=b=1$.

**2130.** If $a,b>0, a^2+b^2=2$ then:

$$\frac{a^2}{b^3+1}+\frac{b^2}{a^3+1}\geq 1$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Tapas Das-India*

$$a^2+b^2=2\ or, 2=\frac{a^2}{1}+\frac{b^2}{1}\overset{Radon}{\geq}\frac{(a+b)^2}{2}\ or\ \ a+b\leq 2\ (1)$$
$$2=a^2+b^2\overset{AM-GM}{\geq}2ab\ or\ ab\leq 1\ (2)$$

$$\frac{a^2}{b^3+1}+\frac{b^2}{a^3+1}=\left(a^2-\frac{a^2b^3}{b^3+1}\right)+\left(b^2-\frac{b^2a^3}{a^3+1}\right)=$$

$$=(a^2+b^2)-a^2b^2\left(\frac{b}{b^3+1}+\frac{a}{a^3+1}\right)\overset{AM-GM}{\geq}(a^2+b^2)-a^2b^2\left(\frac{b}{2b^{\frac{3}{2}}}+\frac{a}{2a^{\frac{3}{2}}}\right)=$$

$$=(a^2+b^2)-\frac{a^2b^2}{2}\left(\sqrt{\frac{1}{b}}+\sqrt{\frac{1}{a}}\right)\overset{\substack{CBS\\ a^2+b^2=2}}{\geq}2-\frac{a^2b^2}{2}\sqrt{2\left(\frac{1}{b}+\frac{1}{a}\right)}=$$
$$=2-\frac{ab\sqrt{ab}}{2}\sqrt{2(a+b)}\overset{(1)\&(2)}{\geq}2-\frac{1}{2}\sqrt{4}=1$$

Equality holds for a=b=1.

**2131.** If $a,b,c>0$ then:

$$4\sum\frac{a^8+b^8}{(a^2+b^2)^2}+abc\sum a\geq\sum a\sum a^3$$

*Proposed by Neculai Stanciu-Romania*

*Solution by Tapas Das-India*

$$We\ know\ that\ by\ Schur's\ inequality\ if\ a,b,c>0\ then:$$
$$a^t(a-b)(a-c)+b^t(b-c)(b-a)+c^t(c-a)(c-b)\geq 0$$

$$For\ t=2\ we\ have\ a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)\geq 0$$

$$\sum(a^4-a^3(b+c)+a^2bc)\ge 0 \text{ or} \sum a^4+abc\sum a\ge\sum a^3(b+c)\ (1)$$

$$a^4+b^4=\frac{(a^2)^2}{1}+\frac{(b^2)^2}{1}\overset{Radon}{\ge}\frac{(a^2+b^2)^2}{2} \text{ or } 2(a^4+b^4)\ge(a^2+b^2)^2\ (2)$$

$$a^8+b^8=\frac{(a^4)^2}{1}+\frac{(b^4)^2}{1}\overset{Radon}{\ge}\frac{(a^4+b^4)^2}{2}\ (3)$$

$$4\sum\frac{a^8+b^8}{(a^2+b^2)^2}+abc\sum a\overset{(3)\&(2)}{\ge}\sum(a^4+b^4)+abc\sum a=2\sum a^4+abc\sum a$$

$$\sum a^4+\left(\sum a^4+abc\sum a\right)\overset{(1)}{\ge}\sum a^4+\sum a^3(b+c)=$$

$$=\sum a^4+\sum a^3(a+b+c-a)=\sum a^4+\sum a^3\sum a-\sum a^4=\sum a\sum a^3$$

Equality holds for a=b=c.

**2132.** **If $a, \mathrm{b}>1$ and $ab=4$ then prove that :**

$$\left(\frac{a-1}{\mathrm{b}}\right)^{\mathrm{b}}\left(\frac{\mathrm{b}-1}{a}\right)^{a}\le\frac{1}{16}$$

***Proposed by Nguyen Hung Cuong-Vietnam***

***Solution by Soumava Chakraborty-Kolkata-India***

$$\text{Let } f(a)=a^a.\left(\frac{4}{a}\right)^{\frac{4}{a}}\ \forall\, a>1 \text{ and then :}$$

$$f'(a)=a^{a-\frac{4}{a}-2}.4^{\frac{4}{a}}.\left((a^2+4)\ln a-8\ln 2+a^2-4\right)$$

$$\text{Now, since } a^{a-\frac{4}{a}-2}.4^{\frac{4}{a}}>0\ \forall\, a>1 \text{ and } (a^2+4)\ln a-8\ln 2+a^2-4\lesseqgtr 0$$

$$\text{according as } a\lesseqgtr 2 \therefore f'(a)\lesseqgtr 0 \text{ according as } a\lesseqgtr 2\Rightarrow f(a) \text{ is}\downarrow \text{on } (1,2]$$

$$\text{and } f(a) \text{ is}\uparrow\text{on } [2,\infty)\Rightarrow f(a)\ge f(2)=16\overset{ab=4}{\Rightarrow}a^a.\mathrm{b}^{\mathrm{b}}\ge 16$$

$$\Rightarrow\left(\frac{a-1}{\mathrm{b}}\right)^{\mathrm{b}}\left(\frac{\mathrm{b}-1}{a}\right)^{a}\le\frac{(a-1)^{\mathrm{b}}(\mathrm{b}-1)^{a}}{16}\overset{?}{\le}\frac{1}{16}\Leftrightarrow \mathrm{b}\ln(a-1)+a\ln(\mathrm{b}-1)\underbrace{\overset{?}{\le}}_{(*)}0$$

$$\text{Since } \ln x\le x-1\ \forall\, x>0 \therefore \text{LHS of } (*)\le \mathrm{b}(a-2)+a(\mathrm{b}-2)\overset{ab=4}{=}8-2(a+\mathrm{b})$$

$$\overset{AM-GM}{\le}8-4.\sqrt{ab}\overset{ab=4}{=}0\Rightarrow(*) \text{ is true} \therefore\left(\frac{a-1}{\mathrm{b}}\right)^{\mathrm{b}}\left(\frac{\mathrm{b}-1}{a}\right)^{a}\le\frac{1}{16}$$

$$\forall\, a,\mathrm{b}>1\,\middle|\, ab=4, ''=''\text{ iff } a=\mathrm{b}=2\ (\mathrm{QED})$$

**2133.** If $a,b,c>0$ then prove that :

$$\frac{a-b}{a^2+1}+\frac{b-c}{b^2+1}+\frac{c-a}{c^2+1}\le 0$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$\boxed{\text{Case 1}}$ $a\ge c\ge b$ or $b\ge c\ge a$ or $b\ge a\ge c$ or $c\ge a\ge b$ and then :

$$\frac{a-b}{a^2+1}+\frac{b-c}{b^2+1}+\frac{c-a}{c^2+1}=\frac{a-c+c-b}{a^2+1}+\frac{b-c}{b^2+1}+\frac{c-a}{c^2+1}$$

$$=(c-a)\left(\frac{1}{c^2+1}-\frac{1}{a^2+1}\right)+(b-c)\left(\frac{1}{b^2+1}-\frac{1}{a^2+1}\right)$$

$$=-\frac{(c-a)^2(c+a)}{(c^2+1)(a^2+1)}+\frac{(b-c)(a-b)(a+b)}{(a^2+1)(b^2+1)}\le 0$$

$\boxed{\text{Case 2}}$ $a\ge b\ge c$ or $c\ge b\ge a$ and then : $\frac{a-b}{a^2+1}+\frac{b-c}{b^2+1}+\frac{c-a}{c^2+1}$

$$=\frac{a-b}{a^2+1}+\frac{b-a+a-c}{b^2+1}+\frac{c-a}{c^2+1}$$

$$=(a-b)\left(\frac{1}{a^2+1}-\frac{1}{b^2+1}\right)+(c-a)\left(\frac{1}{c^2+1}-\frac{1}{b^2+1}\right)$$

$$=-\frac{(a-b)^2(a+b)}{(a^2+1)(b^2+1)}+\frac{(c-a)(b-c)(b+c)}{(a^2+1)(b^2+1)}\le 0$$

Since we have considered all possible cases concerning the ordering of $a,b,c$ amongst each other, hence we conclude : $\frac{a-b}{a^2+1}+\frac{b-c}{b^2+1}+\frac{c-a}{c^2+1}\le 0$ $\forall\, a,b,c>0$, " = " iff $a=b=c$ (QED)

**2134.** If $x,y,z>0,\ x+y+z=xyz$ and $\lambda\le 9$ then :

$$\lambda+x^2y^2z^2\ge\left(3+\frac{\lambda}{9}\right)\left(\sum_{cyc}xy\right)$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$xyz=\sum_{cyc}x\overset{AM-GM}{\ge}3.\sqrt[3]{xyz}\Rightarrow x^2y^2z^2\ge 27\Rightarrow\left(\sum_{cyc}xy\right)^2\ge 3xyz\left(\sum_{cyc}x\right)$$

$$\overset{x+y+z=xyz}{=} 3x^2y^2z^2 \geq 81 \Rightarrow \sum_{cyc} xy \geq 9 \to (1) \text{ and now,}$$

$$\lambda + x^2y^2z^2 \overset{?}{\geq} \left(3 + \frac{\lambda}{9}\right)\left(\sum_{cyc} xy\right) \Leftrightarrow x^2y^2z^2 - 3\sum_{cyc} xy \overset{?}{\geq} \frac{\lambda}{9}\cdot\left(\sum_{cyc} xy - 9\right)$$

$$\text{and now, since } \sum_{cyc} xy - 9 \overset{\text{via }(1)}{\geq} 0 \text{ and } \frac{\lambda}{9} \leq 1 \therefore \text{ it suffices to prove :}$$

$$x^2y^2z^2 - 3\sum_{cyc} xy \overset{?}{\geq} \sum_{cyc} xy - 9 \Leftrightarrow \left(\sum_{cyc} x\right)^2 + \frac{9xyz}{\sum_{cyc} x} - 4\sum_{cyc} xy \overset{?}{\geq} 0$$

$$(\because x + y + z = xyz) \Leftrightarrow \left(\sum_{cyc} x\right)^3 + 9xyz - 4\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow \sum_{cyc} x^3 + 3xyz \overset{?}{\geq} \sum_{cyc} x^2y + \sum_{cyc} xy^2 \to \text{true via Schur}$$

$$\therefore \lambda + x^2y^2z^2 \geq \left(3 + \frac{\lambda}{9}\right)\left(\sum_{cyc} xy\right) \forall x, y, z > 0 \,|\, x + y + z = xyz \text{ and } \lambda \leq 9,$$

$$'' ='' \quad x = y = z = \sqrt{3}$$

**2435. If $a, b, c > 0$ then prove that :**

$$\frac{a^2 - b^2}{b^3 + 1} + \frac{b^2 - c^2}{c^3 + 1} + \frac{c^2 - a^2}{a^3 + 1} \geq 0$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$\boxed{\text{Case 1}}$ $a \geq b \geq c$ or $a \geq c \geq b$ or $b \geq c \geq a$ or $c \geq b \geq a$ and then :

$$\frac{a^2 - b^2}{b^3 + 1} + \frac{b^2 - c^2}{c^3 + 1} + \frac{c^2 - a^2}{a^3 + 1} = \frac{a^2 - c^2 + c^2 - b^2}{b^3 + 1} + \frac{b^2 - c^2}{c^3 + 1} + \frac{c^2 - a^2}{a^3 + 1}$$

$$= (c^2 - a^2)\left(\frac{1}{a^3 + 1} - \frac{1}{b^3 + 1}\right) + (b^2 - c^2)\left(\frac{1}{c^3 + 1} - \frac{1}{b^3 + 1}\right)$$

$$= \frac{(c - a)(b - a)(c + a)(a^2 + ab + b^2)}{(a^3 + 1)(b^3 + 1)} + \frac{(b - c)^2(b + c)(b^2 + bc + c^2)}{(c^3 + 1)(b^3 + 1)} \geq 0$$

$\boxed{\text{Case 2}}$ $b \geq a \geq c$ or $c \geq a \geq b$ and then : $\frac{a^2 - b^2}{b^3 + 1} + \frac{b^2 - c^2}{c^3 + 1} + \frac{c^2 - a^2}{a^3 + 1}$

$$= \frac{a^2 - b^2}{b^3 + 1} + \frac{b^2 - a^2 + a^2 - c^2}{c^3 + 1} + \frac{c^2 - a^2}{a^3 + 1}$$

$$=(a^2-b^2)\left(\frac{1}{b^3+1}-\frac{1}{c^3+1}\right)+(c^2-a^2)\left(\frac{1}{a^3+1}-\frac{1}{c^3+1}\right)$$
$$=\frac{(a-b)(c-b)(a+b)(b^2+bc+c^2)}{(b^3+1)(c^3+1)}+\frac{(c-a)^2(c+a)(c^2+ca+a^2)}{(a^3+1)(c^3+1)}\geq 0$$

Since we have considered *all* possible cases concerning the ordering of $a, b, c$ amongst each other, hence we conclude : $\frac{a^2-b^2}{b^3+1}+\frac{b^2-c^2}{c^3+1}+\frac{c^2-a^2}{a^3+1}\geq 0$ $\forall\, a,b,c>0,''=''$ iff $a=b=c$

**2136.** If $a,b,c>0$ *then prove that* :

$$\frac{a-b}{\sqrt{b+1}}+\frac{b-c}{\sqrt{c+1}}+\frac{c-a}{\sqrt{a+1}}\geq 0$$

***Proposed by Nguyen Hung Cuong-Vietnam***

***Solution by Soumava Chakraborty-Kolkata-India***

$\boxed{\text{Case 1}}$ $a\geq b\geq c$ or $a\geq c\geq b$ or $b\geq c\geq a$ or $c\geq b\geq a$ and then :

$$\frac{a-b}{\sqrt{b+1}}+\frac{b-c}{\sqrt{c+1}}+\frac{c-a}{\sqrt{a+1}}=\frac{a-c+c-b}{\sqrt{b+1}}+\frac{b-c}{\sqrt{c+1}}+\frac{c-a}{\sqrt{a+1}}$$
$$=(c-a)\left(\frac{1}{\sqrt{a+1}}-\frac{1}{\sqrt{b+1}}\right)+(b-c)\left(\frac{1}{\sqrt{c+1}}-\frac{1}{\sqrt{b+1}}\right)$$
$$=\frac{(c-a)(b-a)}{\sqrt{(a+1)(b+1)}.\left(\sqrt{b+1}+\sqrt{a+1}\right)}+\frac{(b-c)^2}{\sqrt{(c+1)(b+1)}.\left(\sqrt{b+1}+\sqrt{c+1}\right)}\geq 0$$

$\boxed{\text{Case 2}}$ $b\geq a\geq c$ or $c\geq a\geq b$ and then : $\frac{a-b}{\sqrt{b+1}}+\frac{b-c}{\sqrt{c+1}}+\frac{c-a}{\sqrt{a+1}}$

$$=\frac{a-b}{\sqrt{b+1}}+\frac{b-a+a-c}{\sqrt{c+1}}+\frac{c-a}{\sqrt{a+1}}$$
$$=(a-b)\left(\frac{1}{\sqrt{b+1}}-\frac{1}{\sqrt{c+1}}\right)+(c-a)\left(\frac{1}{\sqrt{a+1}}-\frac{1}{\sqrt{c+1}}\right)$$
$$=\frac{(a-b)(c-b)}{\sqrt{(b+1)(c+1)}.\left(\sqrt{b+1}+\sqrt{c+1}\right)}+\frac{(c-a)^2}{\sqrt{(a+1)(c+1)}.\left(\sqrt{a+1}+\sqrt{c+1}\right)}\geq 0$$

Since we have considered *all* possible cases concerning the ordering of $a, b, c$ amongst each other, hence we conclude : $\frac{a-b}{\sqrt{b+1}}+\frac{b-c}{\sqrt{c+1}}+\frac{c-a}{\sqrt{a+1}}\geq 0$ $\forall\, a,b,c>0,''=''$ iff $a=b=c$

**2137.** If $a, b, c \geq 0$ and $a + b + c = 1$ then prove that :

$$\frac{16}{27} \leq (1 - 5ab)^2 + (1 - 5bc)^2 + (1 - 5ca)^2 \leq 3$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$\boxed{\text{Case 1}}$ Exactly 2 variables $= 0$ and WLOG we may assume $b = c = 0$ $(a = 1)$ and then : $\sum_{cyc}(1 - 5bc)^2 = 3$ and $\because \frac{16}{27} < 3 = 3 \therefore \frac{16}{27} < \sum_{cyc}(1 - 5bc)^2 = 3$

$\boxed{\text{Case 2}}$ Exactly 1 variable $= 0$ and WLOG we may assume $a = 0$ $\left(\begin{matrix} b + c = 1; \\ b, c > 0\end{matrix}\right)$ and then : $\sum_{cyc}(1 - 5bc)^2 = 2 + (1 - 5bc)^2 \overset{?}{<} 3 \Leftrightarrow 1 - 10bc + 25b^2c^2 \overset{?}{<} 1$

$\Leftrightarrow 2(b + c)^2 \overset{?}{>} 5bc$ $(\because b + c = 1) \Leftrightarrow 2(b - c)^2 + 3bc \overset{?}{>} 0 \to$ true $\because b, c > 0$

$\therefore \sum_{cyc}(1 - 5bc)^2 < 3$ and $\because 2 + (1 - 5bc)^2 > \frac{16}{27} \therefore \sum_{cyc}(1 - 5bc)^2 > \frac{16}{27}$

$\boxed{\text{Case 3}}$ $a, b, c > 0$ and then : $\sum_{cyc}(1 - 5bc)^2 = 3 - 10\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^2 +$

$$25\sum_{cyc} a^2b^2 \ (\because a + b + c = 1) \leq 3 - 30\left(\sum_{cyc} ab\right)^2 + 25\sum_{cyc} a^2b^2$$

$$= 3 - 5\sum_{cyc} a^2b^2 - 60abc\left(\sum_{cyc} a\right) < 3 \ (\because a, b, c > 0) \therefore \boxed{\sum_{cyc}(1 - 5bc)^2 < 3}$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say); then : $\sum_{cyc} a = s, abc = r^2s, \sum_{cyc} ab = 4Rr + r^2, \sum_{cyc} a^2b^2 = r^2((4R + r)^2 - 2s^2)$

and then, $\sum_{cyc}(1 - 5bc)^2 \overset{?}{\geq} \frac{16}{27}$ becomes :

$$3\left(\sum_{cyc} a\right)^4 - 10\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)^2 + 25\sum_{cyc} a^2b^2 \overset{?}{\geq} \frac{16}{27}.\left(\sum_{cyc} a\right)^4$$

$$\Leftrightarrow 65s^4 - 270(4Rr + r^2)s^2 + 675r^2((4R+r)^2 - 2s^2) \boxed{\underset{(*)}{\overset{?}{\geq}}} 0 \text{ and } \because P =$$

$$65(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \overset{?}{\geq} P \Leftrightarrow (100R - 227r)s^2 \boxed{\underset{(**)}{\overset{?}{\geq}}} r(584R^2 - 1580Rr + 95r^2)$$

$$\boxed{\text{Case (i)}}\ 100R - 227r \geq 0 \text{ and then : LHS of } (**) \overset{\text{Gerretsen}}{\geq}$$

$$(100R - 227r)(16Rr - 5r^2) \overset{?}{\geq} \text{RHS of } (**) \Leftrightarrow 8r(R - 2r)(172R - 65r) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (**) \text{ is true}$$

$$\boxed{\text{Case (ii)}}\ 100R - 227r < 0 \text{ and then : LHS of } (**) \overset{\text{Gerretsen}}{\geq}$$

$$(100R - 227r)(4R^2 + 4Rr + 3r^2) \overset{?}{\geq} \text{RHS of } (**)$$

$$\Leftrightarrow 100t^3 - 273t^2 + 243t - 194 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(100t^2 - 73t + 97) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \text{ is true} \therefore \text{combining both cases}, (**) \Rightarrow (*) \text{ is true}$$

$$\forall \Delta XYZ_{s,R,r} \therefore \boxed{\sum_{\text{cyc}} (1 - 5bc)^2 \geq \frac{16}{27}} \therefore \text{combining cases } 1, 2 \text{ and } 3,$$

$$\frac{16}{27} \leq \sum_{\text{cyc}} (1 - 5bc)^2 \leq 3, \text{"="} \text{ for lower bound iff } a = b = c = \frac{1}{3} \text{ and}$$

$$\text{"="} \text{ for upper bound iff } \binom{a = 1}{b = c = 0} \text{ or } \binom{b = 1}{a = c = 0} \text{ or } \binom{c = 1}{a = b = 0} \text{ (QED)}$$

**2138.** If $a, b, c > 0, abc = 1$ and $\lambda \geq 1$ then :

$$\sum_{\text{cyc}} \frac{a+b}{a+b+\lambda} \geq \frac{6}{\lambda+2}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{\text{cyc}} \frac{a+b}{a+b+\lambda} = \sum_{\text{cyc}} \frac{x+\lambda-\lambda}{x+\lambda} \ (x = b+c, y = c+a, z = a+b)$$

$$= 3 - \frac{\lambda\left(\sum_{\text{cyc}} xy + 3\lambda^2 + 2\lambda \sum_{\text{cyc}} x\right)}{xyz + \lambda^3 + \lambda \sum_{\text{cyc}} xy + \lambda^2 \sum_{\text{cyc}} x} \overset{?}{\geq} \frac{6}{\lambda+2}$$

$$\Leftrightarrow 3xyz + 2(\lambda - 1)\left(\sum_{\text{cyc}} xy\right) + (\lambda^2 - 4\lambda)\left(\sum_{\text{cyc}} x\right) - 6\lambda^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow 3\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)-3+2(\lambda-1)\left(\left(\sum_{cyc} a\right)^2+\sum_{cyc} ab\right)+$$

$$(2\lambda^2-8\lambda)\left(\sum_{cyc} a\right)-6\lambda^2 \boxed{\overset{?}{\underset{(*)}{\geq}}} 0$$

$$\text{Now, LHS of } (*) \overset{AM-GM}{\geq} 2\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)+9abc-3+$$

$$2(\lambda-1)\left(\left(\sum_{cyc} a\right)^2+3\right)+2\lambda^2\left(\sum_{cyc} a-3\right)-8\lambda\left(\sum_{cyc} a\right)$$

$$\left(\because \sum_{cyc} ab \overset{AM-GM}{\geq} 3.\sqrt[3]{a^2b^2c^2}=3\right) \overset{\substack{\lambda\geq 1\\ and\\ \sum_{cyc} ab\geq 3}}{\geq}$$

$$6t+9-3+2(\lambda-1)(t^2+3)+2(t-3)-8\lambda t \left(\begin{array}{c} t=\sum_{cyc} a \text{ and } \because abc=1 \text{ and} \\ \because t=\sum_{cyc} a \overset{AM-GM}{\geq} 3.\sqrt[3]{abc}=3 \end{array}\right)$$

$$=8t-8\lambda t+9-3-6+2(\lambda-1)(t^2+3)=2(\lambda-1)(t^2+3-4t)$$
$$=2(\lambda-1)(t-3)(t-1)\geq 0(\because t\geq 3 \text{ and } \lambda\geq 1)\Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{a+b}{a+b+\lambda}\geq \frac{6}{\lambda+2}\forall a,b,c>0\,\middle|\, abc=1 \text{ and } \lambda\geq 1,$$

$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

**2139. If $a,b,c\in[0,1]$ and $\lambda\geq 7$ then :**

$$\sum_{cyc}\frac{a}{b+c^3+\lambda}\leq\frac{3}{\lambda+2}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$a,b,c\in[0,1]\Rightarrow b\geq b^3 \text{ and analogs} \therefore \sum_{cyc}\frac{a}{b+c^3+\lambda}\leq\sum_{cyc}\frac{a}{b^3+c^3+\lambda}$$

$$=\sum_{cyc}\frac{a}{\lambda+2-(1-b^3)-(1-c^3)}=\sum_{cyc}\frac{\sqrt[3]{1-x}}{m-y-z}$$

$$(x=1-a^3, y=1-b^3, z=1-c^3, m=\lambda+2)\leq\sum_{cyc}\frac{\frac{3-x}{3}}{m-y-z}$$

$$\left(\begin{array}{c}\because\sqrt[3]{1-x}\overset{?}{\leq}\frac{3-x}{3}\Leftrightarrow(3-x)^3\overset{?}{\geq}27(1-x)\Leftrightarrow x^2(9-x)\overset{?}{\geq}0\\ \to\text{true}\because 0\leq a^3,b^3,c^3\leq 1\Rightarrow 0\leq x,y,z\leq 1\therefore\sqrt[3]{1-x}\leq\frac{3-x}{3}\text{ and analogs}\end{array}\right)$$

$$\overset{?}{\leq}\frac{3}{\lambda+2}=\frac{3}{m}\Leftrightarrow\sum_{cyc}\frac{m(3-x)}{m-y-z}\boxed{\overset{?}{\leq}(*)}9$$

$$\text{Now, }\sum_{cyc}\frac{m(3-x)}{m-y-z}=\sum_{cyc}\frac{(m-y-z+y+z)(3-x)}{m-y-z}$$

$$=\sum_{cyc}(3-x)+\sum_{cyc}\frac{(y+z)(3-x)}{m-y-z}\leq 9-\sum_{cyc}x+\sum_{cyc}\frac{(y+z)(3-x)}{7}$$

$$\left(\begin{array}{c}\because 0\leq y,z\leq 1\Rightarrow m-y-z\geq m-2\geq 9-2=7\text{ as }m=\lambda+2\geq 9\\ \text{and}\because 0\leq x,y,z\leq 1\Rightarrow y+z\geq 0\text{ and }3-x>0\end{array}\right)$$

$$\leq 9-\sum_{cyc}x+\frac{1}{7}.\left(6\sum_{cyc}x-2\sum_{cyc}xy\right)=9-\frac{1}{7}.\left(\sum_{cyc}x+2\sum_{cyc}xy\right)\leq 9$$

$$(\because x,y,z\geq 0)\Rightarrow(*)\text{ is true}\therefore\sum_{cyc}\frac{a}{b+c^3+\lambda}\leq\frac{3}{\lambda+2}\ \forall\, a,b,c\in[0,1]\text{ and }\lambda\geq 7,$$

$$''=''\text{ iff }(a=a^3\text{ and analogs})\wedge(x=1-a^3=y=1-b^3=z=1-c^3=0)$$
$$\Rightarrow''=''\text{ iff }a=b=c=1\text{ (QED)}$$

**2140.** If $a,b,c>0, a^2+b^2+c^2=3$ and $\lambda\geq 0$ then :

$$\sum_{cyc}\left(a.\sqrt{\lambda+b+c}\right)\leq\sqrt{\lambda+2}(a+b+c)$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc}\left(a.\sqrt{\lambda+b+c}\right)=\sum_{cyc}\left(\sqrt{a}.\sqrt{\lambda a+ab+ca}\right)\overset{CBS}{\leq}$$

$$\leq\sqrt{\sum_{cyc}a}.\sqrt{\lambda\sum_{cyc}a+2\sum_{cyc}ab}\overset{?}{\leq}\sqrt{\lambda+2}.\left(\sum_{cyc}a\right)$$

$$\Leftrightarrow \lambda\left(\sum_{cyc} a\right)^2 + 2\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \overset{?}{\le} \lambda\left(\sum_{cyc} a\right)^2 + 2\left(\sum_{cyc} a\right)^2 \Leftrightarrow \sum_{cyc} a \overset{?}{\ge} \sum_{cyc} ab$$

$$\Leftrightarrow \left(\sum_{cyc} a\right).\sqrt{\frac{1}{3}\sum_{cyc} a^2} \overset{?}{\ge} \sum_{cyc} ab \left(\because \sum_{cyc} a^2 = 3\right)$$

$$\Leftrightarrow \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a\right)^2 \overset{?}{\ge} 3\left(\sum_{cyc} ab\right)^2 \to \text{true} \because \sum_{cyc} a^2 \ge \sum_{cyc} ab \text{ and}$$

$$\left(\sum_{cyc} a\right)^2 \ge 3\sum_{cyc} ab \therefore \sum_{cyc}\left(a.\sqrt{\lambda + b + c}\right) \le \sqrt{\lambda + 2}.(a + b + c)$$

$$\forall\, a, b, c > 0 \,\middle|\, a^2 + b^2 + c^2 = 3 \text{ and } \lambda \ge 0, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

**2141. If $x, y, z > 0, xyz = 1$ and $\lambda \ge 0$ then :**

$$\sum_{cyc} \sqrt{x^2 - xy + y^2} \le \lambda\left(\sum_{cyc} x^2\right)^2 + (1 - 3\lambda)\left(\sum_{cyc} x^2\right)$$

***Proposed by Marin Chirciu-Romania***

***Solution by Soumava Chakraborty-Kolkata-India***

$$\sqrt{\left(\sum_{cyc} xy\right)\left(2\sum_{cyc} x^2 - \sum_{cyc} xy\right)} \overset{AM-GM}{\le} \sum_{cyc} x^2$$

$$\Rightarrow \sum_{cyc} x^2 \ge \sqrt{\frac{\sum_{cyc} xy}{3}}.\sqrt{\sum_{cyc}(1)\left(\sum_{cyc}(x^2 - xy + y^2)\right)} \overset{\text{Reverse CBS}}{\ge}$$

$$\sqrt{\frac{\sum_{cyc} xy}{3}}.\sum_{cyc} \sqrt{x^2 - xy + y^2} \ge \sum_{cyc} \sqrt{x^2 - xy + y^2}$$

$$\left(\because \sum_{cyc} xy \overset{AM-GM}{\ge} 3.\sqrt[3]{x^2y^2z^2} \overset{xyz=1}{=} 3\right)$$

$$\therefore \sum_{cyc} x^2 + \lambda\left(\sum_{cyc} x^2\right)^2 \ge \sum_{cyc}\sqrt{x^2 - xy + y^2} + \lambda\left(\sum_{cyc} x^2\right)^2 \ge$$

$$\sum_{cyc}\sqrt{x^2 - xy + y^2} + 3\lambda\left(\sum_{cyc} x^2\right)\left(\because \sum_{cyc} x^2 \overset{AM-GM}{\ge} 3.\sqrt[3]{x^2y^2z^2} \overset{xyz=1}{=} 3 \atop \text{and} \because \lambda \ge 0\right)$$

$$\therefore \sum_{cyc}\sqrt{x^2 - xy + y^2} \le \lambda\left(\sum_{cyc} x^2\right)^2 + (1 - 3\lambda)\left(\sum_{cyc} x^2\right)$$

$$\forall\, x,y,z > 0 \,\big|\, xyz = 1 \text{ and } \lambda \ge 0, '' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$

**2142. If $x, y, z > 0, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2$ and $27\lambda + 8n = 36, \lambda \ge 1, n > 0$ then :**

$$\sum_{cyc} x \le \lambda xyz + n$$

***Proposed by Marin Chirciu-Romania***

***Solution by Soumava Chakraborty-Kolkata-India***

$$2 = \sum_{cyc}\frac{1}{x} \overset{AM-GM}{\ge} \frac{3}{\sqrt[3]{xyz}} \Rightarrow 8xyz \ge 27 \to (1) \text{ and now, } \sum_{cyc} x \overset{?}{\le} \lambda xyz + n$$

$$\Leftrightarrow 8\sum_{cyc} x \overset{?}{\le} 8\lambda xyz + 36 - 27\lambda \;(\because 27\lambda + 8n = 36)$$

$$\Leftrightarrow 8\sum_{cyc} x \overset{?}{\le} \lambda(8xyz - 27) + 36 \text{ and} \because 8xyz - 27 \overset{\text{via }(1)}{\ge} 0 \text{ and} \because \lambda \ge 1$$

$$\therefore \text{it suffices to prove : } 8\sum_{cyc} x \overset{?}{\le} 8xyz - 27 + 36$$

$$\Leftrightarrow 8\sum_{cyc} x \overset{?}{\le} 8xyz\left(\frac{\sum_{cyc} xy}{2xyz}\right)^2 + 9\left(\frac{2xyz}{\sum_{cyc} xy}\right)\left(\because \frac{\sum_{cyc} xy}{2xyz} = 1\right)$$

$$\Leftrightarrow 4xyz\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \overset{?}{\le} \left(\sum_{cyc} xy\right)^3 + 9x^2y^2z^2$$

$$\Leftrightarrow \sum_{cyc} x^3y^3 + 3x^2y^2z^2 \overset{?}{\geq} xyz\left(\sum_{cyc} x^2y + \sum_{cyc} xy^2\right) \to \text{true}$$

$$\because \sum_{cyc} a^3 + 3abc \overset{\text{Schur}}{\geq} \sum_{cyc} a^2b + \sum_{cyc} ab^2 \text{ and choosing } a \equiv xy, b \equiv yz, c \equiv zx$$

$$\therefore \sum_{cyc} x \leq \lambda xyz + n \ \forall x, y, z > 0 \,\Big|\, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2 \text{ and } 27\lambda + 8n = 36, \lambda \geq 1, n > 0,$$

$$'' ='' \text{ iff } x = y = z = \frac{3}{2} \text{ (QED)}$$

**2143.** If $x, y, z > 0, xy + yz + zx = 2$ and $\lambda \geq \frac{8}{9}$ then :

$$\lambda xyz(x + y + z) - x^2y^2z^2 \leq \frac{4}{27}(9\lambda - 2)$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$xyz\left(\sum_{cyc} x\right) \leq \frac{1}{3}\left(\sum_{cyc} xy\right)^2 = \frac{4}{3} \Rightarrow \frac{4}{3} - xyz\left(\sum_{cyc} x\right) \geq 0 \to (1) \text{ and now,}$$

$$xyz\left(\sum_{cyc} x\right) - x^2y^2z^2 \overset{?}{\leq} \frac{4}{27}(9\lambda - 2) \Leftrightarrow \lambda\left(\frac{4}{3} - xyz\left(\sum_{cyc} x\right)\right) - \frac{8}{27} + x^2y^2z^2 \overset{?}{\geq} 0$$

$$\text{and } \because \frac{4}{3} - xyz\left(\sum_{cyc} x\right) \overset{\text{via } (1)}{\geq} 0 \text{ and } \because \lambda \geq \frac{8}{9} \therefore \text{it suffices to prove :}$$

$$\frac{8}{9}\left(\frac{4}{3} - xyz\left(\sum_{cyc} x\right)\right) - \frac{8}{27} + x^2y^2z^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{4}{9}\left(\sum_{cyc} xy\right)\left(\frac{1}{3}\left(\sum_{cyc} xy\right)^2 - xyz\left(\sum_{cyc} x\right)\right) - \frac{1}{27}\left(\sum_{cyc} xy\right)^3 + x^2y^2z^2 \overset{?}{\geq} 0$$

$$\left(\because \sum_{cyc} xy = 2\right) \Leftrightarrow \left(\sum_{cyc} xy\right)^3 + 9x^2y^2z^2 \overset{?}{\geq} 4xyz\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)$$

$$\Leftrightarrow \sum_{cyc} x^3y^3 + 3x^2y^2z^2 \overset{?}{\geq} xyz\left(\sum_{cyc} x^2y + \sum_{cyc} xy^2\right) \to \text{true}$$

$$\because \sum_{cyc} a^3 + 3abc \overset{\text{Schur}}{\geq} \sum_{cyc} a^2b + \sum_{cyc} ab^2 \text{ and choosing } a \equiv xy, b \equiv yz, c \equiv zx$$

$$\therefore \lambda xyz(x+y+z) - x^2y^2z^2 \leq \frac{4}{27}(9\lambda - 2)\ \forall\, x,y,z > 0 \,|\, xy + yz + zx = 2 \text{ and}$$

$$\lambda \geq \frac{8}{9}, '' ='' \text{ iff } x = y = z = \sqrt{\frac{2}{3}}\ (\text{QED})$$

**2144. If $a, b, c > 0$ then:**

$$\sum \frac{a^{12}}{a^{12} + (b^3 + c^3)a^3b^3c^3} \geq 1$$

***Proposed by Neculai Stanciu, Mihaly Bencze-Romania***

***Solution by Tapas Das-India***

$$We\ know\ that:$$

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2) \overset{AM-GM}{\geq} (x+y)(2xy - xy) = xy(x+y)$$

$$b^3c^3(b^3 + c^3) \leq (b^3)^3 + (c^3)^3 = b^9 + c^9$$

$$a^{12} + a^3\, b^3c^3(b^3 + c^3) \leq a^{12} + a^3(b^9 + c^9) = a^3(a^9 + b^9 + c^9)$$

$$\sum \frac{a^{12}}{a^{12} + (b^3 + c^3)a^3b^3c^3} \geq \sum \frac{a^{12}}{a^3(a^9 + b^9 + c^9)} = \sum \frac{a^9}{(a^9 + b^9 + c^9)} = 1$$

Equality holds for a=b=c.

**2145. If $a, b > 0, a + b = 2$ then:**

$$\frac{a}{b^2 + 1} + \frac{b}{a^2 + 1} \geq 1$$

***Proposed by Nguyen Hung Cuong-Vietnam***

***Solution by Daniel Sitaru-Romania***

**Denote:**

$$\begin{cases} S = x + y \\ P = xy \end{cases}$$

$$2=a+b\overset{AM-GM}{\geq}2\sqrt{ab}=2\sqrt{P}\Rightarrow 2\geq 2\sqrt{P}\Rightarrow P\leq 1$$

$$P-1\leq 0\quad (1)$$

$$P>0\Rightarrow P+5>0\qquad (2)$$

$$\frac{a}{b^2+1}+\frac{b}{a^2+1}=\frac{a(a^2+1)+b(b^2+1)}{(a^2+1)(b^2+1)}=$$

$$=\frac{a^3+b^3+a+b}{a^2b^2+a^2+b^2+1}=\frac{S^3-3SP+S}{P^2+S^2-2P+1}=\frac{8-6P+2}{P^2+4-2P+1}=\frac{10-6P}{P^2-2P+5}\geq 1\Leftrightarrow$$

$$10-6P\geq P^2-2P+5$$

$$P^2+4P-5\leq 0$$

By $(1),(2)$:

$$(P-1)(P+5)\leq 0$$

Equality holds for $a=b=1$.

**2146.** If $x,y,z>0$ *then prove that* :

$$\frac{(xy)^3}{z^9+1}+\frac{(yz)^3}{x^9+1}+\frac{(zx)^3}{y^9+1}\geq\frac{3(xyz)^2}{(xyz)^3+1}$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Let } f(t)=\frac{1}{t^{11}+t^2}\ \forall\, t>0 \text{ and then : } f''(t)=\frac{6(22t^{18}-4t^9+1)}{t^4(t^9+1)^3}$$
$$=\frac{6(18t^{18}+(2t^9-1)^2)}{t^4(t^9+1)^3}>0\ \therefore f(t)\text{ is convex and so,}$$

$$\text{via Weighted Jensen's Inequality, }\frac{(xy)^3}{z^9+1}+\frac{(yz)^3}{x^9+1}+\frac{(zx)^3}{y^9+1}=(xyz)^2.\sum_{cyc}\frac{yz}{x^{11}+x^2}$$

$$\geq (xyz)^2.\left(\sum_{cyc}yz\right).\frac{1}{\left(\frac{\sum_{cyc}(yz.x)}{\sum_{cyc}yz}\right)^{11}+\left(\frac{\sum_{cyc}(yz.x)}{\sum_{cyc}yz}\right)^{2}}\overset{AM-GM}{\geq}$$

$$(xyz)^2.\left(3.(xyz)^{\frac{2}{3}}\right).\frac{1}{\left(\frac{3xyz}{3.(xyz)^{\frac{2}{3}}}\right)^{11}+\left(\frac{3xyz}{3.(xyz)^{\frac{2}{3}}}\right)^{2}}=\frac{3(xyz)^2.(xyz)^{\frac{2}{3}}}{(xyz)^{\frac{11}{3}}+(xyz)^{\frac{2}{3}}}=\frac{3(xyz)^2}{(xyz)^3+1}$$

$$and\ \text{so},\frac{(xy)^3}{z^9+1}+\frac{(yz)^3}{x^9+1}+\frac{(zx)^3}{y^9+1}\geq\frac{3(xyz)^2}{(xyz)^3+1}\ \forall\ x,y,z>0,$$

$$''=''\ \text{iff}\ x=y=z\ (\text{QED})$$

**2147.** If $a,b,c>0$ then:

$$\frac{b-a}{\sqrt{b+1}}+\frac{c-b}{\sqrt{c+1}}+\frac{a-c}{\sqrt{a+1}}\leq 0$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Tapas Das-India*

$$Let\ a+1=x^2,b+1=y^2,c+1=z^2$$

$$then\ b-a=y^2-x^2,c-b=z^2-y^2,a-c=x^2-z^2$$

$$\frac{b-a}{\sqrt{b+1}}+\frac{c-b}{\sqrt{c+1}}+\frac{a-c}{\sqrt{a+1}}=\frac{y^2-x^2}{y}+\frac{z^2-y^2}{z}+\frac{x^2-z^2}{x}=$$

$$=(x+y+z)-\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)\overset{Bergstrom}{\leq}(x+y+z)-\frac{(x+y+z)^2}{x+y+z}=0$$

Equality holds for a=b=c.

**2148.** If $x,y,z\geq 0$ and $x+y+z=1$ then prove that :

$$\frac{1}{x^3+y^3+z^3}+\frac{3}{xyz}\geq 90$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Assigning } y+z=a,z+x=b,x+y=c\Rightarrow a+b-c=2z>0,b+c-a$$
$$=2x>0\ and\ c+a-b=2y>0\Rightarrow a+b>c,b+c>a,c+a>b$$
$$\Rightarrow a,b,c \text{ form sides of } a \text{ triangle with semiperimeter, circumradius and inradius}$$
$$=s,R,r\ (say)\Rightarrow\sum_{cyc}x=s,xyz=r^2s\ and\ \sum_{cyc}x^3=s^3-12Rrs$$

$$\therefore \frac{1}{x^3+y^3+z^3}+\frac{3}{xyz}\overset{?}{\geq}90 \Leftrightarrow \frac{\left(\sum_{cyc}x\right)^3}{\sum_{cyc}x^3}+\frac{3}{xyz}\left(\sum_{cyc}x\right)^3\overset{?}{\geq}90$$

$$\Leftrightarrow xyz\left(\sum_{cyc}x\right)^3+3\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}x\right)^3\overset{?}{\geq}90xyz\left(\sum_{cyc}x^3\right)$$

$$\Leftrightarrow r^2s.s^3+3s^3(s^3-12Rrs)\overset{?}{\geq}90r^2s(s^3-12Rrs)$$

$$\Leftrightarrow 3s^4-(36Rr+89r^2)s^2+1080Rr^3\boxed{\underset{(*)}{\overset{?}{\geq}}}0$$

$$\text{Now, since }(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\ \therefore \text{ in order to prove }(*),$$

$$\text{it suffices to prove}: \text{LHS of }(*)\overset{?}{\geq}3(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (60R-119r)s^2\boxed{\underset{(**)}{\overset{?}{\geq}}}r(768R^2-1560Rr+75r^2)\ \textit{and finally},$$

$$(60R-119r)(16Rr-5r^2)\overset{?}{\geq}r(768R^2-1560Rr+75r^2)$$

$$\Leftrightarrow 4r(R-2r)(48R-65r)\overset{?}{\geq}0\to \text{true}\because R\overset{\text{Euler}}{\geq}2r\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore \frac{1}{x^3+y^3+z^3}+\frac{3}{xyz}\geq 90\ \forall\, x,y,z>0\,\middle|\,x+y+z=1,$$

$$''=''\ \text{ iff }\ x=y=z=\frac{1}{3}\ (\text{QED})$$

**2149.** **If $x,y,z>0$ and $x+y+z=xyz$ then prove that :**

$$\sum_{cyc}\left(x(1+yz)\sqrt{1+x^2}\right)\geq 24\sqrt{3}$$

***Proposed by Zaza Mzhavanadze-Georgia***

***Solution by Soumava Chakraborty-Kolkata-India***

$$\sum_{cyc}\left(x(1+yz).\sqrt{1+x^2}\right)=\sum_{cyc}\left((x+xyz).\sqrt{\frac{xyz}{x+y+z}+x^2}\right)$$

$$\left(\because 1=\frac{xyz}{x+y+z}\right)=\sum_{cyc}\left((x+x+y+z).\sqrt{\frac{x}{x+y+z}.(yz+x^2+xy+xz)}\right)$$

$$(\because xyz = x+y+z) = \sum_{cyc}\left(((x+y)+(z+x)).\sqrt{\frac{x}{xyz}.(x+y)(z+x)}\right)$$

$$(\because x+y+z = xyz) \overset{AM-GM}{\geq} \sum_{cyc}\left(2.\sqrt{(x+y)(z+x)}.\frac{\sqrt{(x+y)(z+x)}}{\frac{y+z}{2}}\right)$$

$$= 4\sum_{cyc}\frac{(x+y)(z+x)}{y+z} \overset{AM-GM}{\geq} 12.\sqrt[3]{(x+y)(y+z)(z+x)} \overset{Cesaro}{\geq} 12.\sqrt[3]{8xyz} \overset{xyz\geq 3\sqrt{3}}{\geq}$$

$$12.\sqrt[3]{8(3\sqrt{3})}\left(\because xyz = \sum_{cyc}x \overset{AM-GM}{\geq} 3.\sqrt[3]{xyz} \Rightarrow \sqrt[3]{x^2y^2z^2} \geq 3 \Rightarrow xyz \geq 3\sqrt{3}\right)$$

$$= 24\sqrt{3} \therefore \sum_{cyc}\left(x(1+yz).\sqrt{1+x^2}\right) \geq 24\sqrt{3}\ \forall\ x,y,z > 0 \,|\, x+y+z = xyz,$$

$$'' ='' \text{ iff } x = y = z = \sqrt{3}\ (QED)$$

**2150. If $a, b, c > 0, a+b+c = 1$ and $\lambda \geq 2$ then :**

$$\sum_{cyc} a^2 \leq (\lambda+1)\sqrt{\frac{\sum_{cyc} a^2}{3}} - \lambda\sqrt{3abc}$$

***Proposed by Marin Chirciu-Romania***

***Solution by Soumava Chakraborty-Kolkata-India***

$$1 = \sum_{cyc} a \overset{AM-GM}{\geq} 3.\sqrt[3]{abc} \Rightarrow abc \leq \frac{1}{27} \Rightarrow \sqrt{3abc} \leq \frac{1}{3} \to (1) \text{ and also,}$$

$$\sqrt{\frac{\sum_{cyc} a^2}{3}} \geq \sqrt{\frac{\left(\sum_{cyc} a\right)^2}{9}} \Rightarrow \sqrt{\frac{\sum_{cyc} a^2}{3}} \geq \frac{1}{3} \to (2)\left(\because \sum_{cyc} a = 1\right) \therefore (1) \text{ and } (2) \Rightarrow$$

$$\sqrt{\frac{\sum_{cyc} a^2}{3}} - \sqrt{3abc} \geq 0 \to (3) \text{ and now, } (\lambda+1).\sqrt{\frac{\sum_{cyc} a^2}{3}} - \lambda.\sqrt{3abc}$$

$$= \lambda\left(\sqrt{\frac{\sum_{cyc} a^2}{3}} - \sqrt{3abc}\right) + \sqrt{\frac{\sum_{cyc} a^2}{3}} \overset{\substack{\lambda\geq 2 \\ \text{and} \\ \text{via } (3)}}{\geq} 2\left(\sqrt{\frac{\sum_{cyc} a^2}{3}} - \sqrt{3abc}\right) + \sqrt{\frac{\sum_{cyc} a^2}{3}}$$

$$\overset{a+b+c=1}{=}\sqrt{3\sum_{cyc}a^2-2.\sqrt{3abc\left(\sum_{cyc}a\right)}}\geq\sqrt{3\sum_{cyc}a^2-2\sum_{cyc}ab}$$

$$=\sqrt{3\sum_{cyc}a^2-\left(2\sum_{cyc}ab+\sum_{cyc}a^2\right)+\sum_{cyc}a^2}\geq\sum_{cyc}a-\left(\sum_{cyc}a\right)^2+\sum_{cyc}a^2$$

$$\overset{a+b+c=1}{=}\sum_{cyc}a^2\text{ and so, }\sum_{cyc}a^2\leq(\lambda+1).\sqrt{\frac{\sum_{cyc}a^2}{3}}-\lambda.\sqrt{3abc}$$

$$\forall\, a,b,c>0\,\middle|\, a+b+c=1\text{ and }\lambda\geq 2,''=''\text{ iff }a=b=c=\frac{1}{3}\text{ (QED)}$$

**2151. If $a,b,c>0, a+b+c=1$ and $\lambda\geq\frac{6}{5}$ then :**

$$\sum_{cyc}a^2\leq\frac{3-\lambda}{9}+\lambda\sum_{cyc}a^3$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc}a^2\overset{?}{\leq}\frac{3-\lambda}{9}+\lambda\sum_{cyc}a^3\overset{a+b+c=1}{\Leftrightarrow}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)\overset{?}{\leq}$$

$$\frac{3-\lambda}{9}.\left(\sum_{cyc}a\right)^3+\lambda\sum_{cyc}a^3$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)\overset{?}{\leq}\frac{1}{3}.\left(\sum_{cyc}a\right)^3+\frac{\lambda}{9}\left(9\sum_{cyc}a^3-\left(\sum_{cyc}a\right)^3\right)\text{ and }\because$$

$$9\sum_{cyc}a^3-\left(\sum_{cyc}a\right)^3\overset{Holder}{\geq}0\text{ and }\because\lambda\geq\frac{6}{5}\therefore\text{ it suffices to prove :}$$

$$\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)\overset{?}{\leq}\frac{1}{3}.\left(\sum_{cyc}a\right)^3+\frac{\left(\frac{6}{5}\right)}{9}\left(9\sum_{cyc}a^3-\left(\sum_{cyc}a\right)^3\right)$$

$$\Leftrightarrow 6\sum_{cyc} a^3 + \left(\sum_{cyc} a\right)^3 - 5\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow \sum_{cyc} a^3 + 3abc \overset{?}{\geq} \sum_{cyc} a^2b + \sum_{cyc} ab^2 \to \text{true via Schur} \therefore \sum_{cyc} a^2 \leq \frac{3-\lambda}{9} + \lambda \sum_{cyc} a^3$$

$$\forall\, a,b,c > 0 \,\middle|\, a+b+c = 1 \text{ and } \lambda \geq \frac{6}{5}, ''='' \text{ iff } a = b = c = \frac{1}{3}.$$

**2152.** If $a,b,c > 0$ then:

$$\sum \frac{a}{b+c} - \frac{4abc}{2abc + \sum ab(a+b)} \geq 1$$

*Proposed by Neculai Stanciu-Romania*

*Solution by Tapas Das-India*

$$2abc + \sum ab(a+b) = 2abc + (a+b+c)(ab+bc+ca) - 3abc =$$

$$= (a+b+c)(ab+bc+ca) - abc =$$

$$= (a+b)(b+c)(c+a) \overset{Cesaro}{\geq} 8abc\ (1)$$

$$\sum \frac{a}{b+c} - \frac{4abc}{2abc + \sum ab(a+b)} \overset{Nesbitt\ \&\ (1)}{\geq} \frac{3}{2} - \frac{4abc}{8abc} = 1$$

Equality holds for a=b=c.

**2153.** If $a,b,c > 0, a+b+c = 1$ and $0 \leq \lambda \leq 4$ then :

$$\lambda \sum_{cyc} ab \leq 1 + 9(\lambda - 3)abc$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\lambda \sum_{cyc} ab \overset{?}{\leq} 1 + 9(\lambda - 3)abc$$

$$\Leftrightarrow \lambda\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc\right) \overset{?}{\leq} \left(\sum_{cyc} a\right)^3 - 27abc \text{ and } \because 0 \leq \lambda \leq 4$$

$$\text{and } \left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) \overset{AM-GM}{\geq} 9abc \therefore \lambda\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc\right) \leq$$

$$4\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc\right) \overset{?}{\leq} \left(\sum_{cyc} a\right)^3 - 27abc$$

$$\Leftrightarrow \sum_{cyc} a^3 + 3abc \overset{?}{\geq} \sum_{cyc} a^2b + \sum_{cyc} ab^2 \to \text{true via Schur}$$

$$\therefore \lambda \sum_{cyc} ab \leq 1 + 9(\lambda - 3)abc \,\forall\, a, b, c > 0 \,\middle|\, a + b + c = 1 \text{ and } 0 \leq \lambda \leq 4,$$

$$'' ='' \text{ iff } a = b = c = \frac{1}{3} \text{ (QED)}$$

**2154. If $x, y, z > 0, xyz \geq 1$ and $n \in \mathbb{N}^*$ then :**

$$\sum_{cyc} \frac{x^{9n-1}}{x^2 + 1} \geq \frac{3}{2}(xyz)^{3n-1}$$

***Proposed by Marin Chirciu-Romania***

***Solution by Soumava Chakraborty-Kolkata-India***

$$\sum_{cyc} \frac{x^{9n-1}}{x^2+1} = \sum_{cyc} \frac{x^{9n-3}.(x^2+1-1)}{x^2+1} = \sum_{cyc} x^{9n-3} - \sum_{cyc} \frac{x^{9n-3}}{x^2+1}$$

$$\overset{AM-GM}{\geq} \frac{1}{2}\sum_{cyc} x^{9n-3} + \frac{1}{2}\sum_{cyc} x^{9n-3} - \sum_{cyc} \frac{x^{9n-3}}{2x} \overset{AM-GM}{\geq}$$

$$\frac{3}{2}.\sqrt[3]{\prod_{cyc} x^{9n-3}} + \frac{1}{2}\sum_{cyc} x^{9n-4}.x - \frac{1}{2}\sum_{cyc} x^{9n-4}$$

$$\overset{Chebyshev}{\geq} \frac{3}{2}(xyz)^{3n-1} + \frac{1}{6}\left(\sum_{cyc} x^{9n-4}\right)\left(\sum_{cyc} x\right) - \frac{1}{2}\sum_{cyc} x^{9n-4}$$

$$\left(\because \text{ WLOG assuming } x \geq y \geq z \Rightarrow x^{9n-4} \geq y^{9n-4} \geq z^{9n-4} \text{ as } \atop n \in \mathbb{N}^* \Rightarrow (9n-4) > 0\right)$$

$$\overset{AM-GM}{\geq} \frac{3}{2}(xyz)^{3n-1} + \frac{1}{6}\left(\sum_{cyc} x^{9n-4}\right)(3.\sqrt[3]{xyz}) - \frac{1}{2}\sum_{cyc} x^{9n-4}$$

$$\overset{xyz \geq 1}{\geq} \frac{3}{2}(xyz)^{3n-1} + \frac{1}{2}\sum_{cyc} x^{9n-4} - \frac{1}{2}\sum_{cyc} x^{9n-4} = \frac{3}{2}(xyz)^{3n-1} \text{ and so,}$$

$$\sum_{cyc} \frac{x^{9n-1}}{x^2+1} \geq \frac{3}{2}(xyz)^{3n-1} \ \forall\, x, y, z > 0 \,\middle|\, xyz \geq 1 \text{ and } n \in \mathbb{N}^*,$$

$$'' ='' \text{ iff } x = y = z = 1 \text{ (QED)}$$

**2155.**

**If $a, b > 0$ and $a^2 + b^2 = 2$ then prove that :**

$$\frac{a^2}{\sqrt{a+3}} + \frac{b^2}{\sqrt{b+3}} \leq 1$$

***Proposed by Nguyen Hung Cuong-Vietnam***

***Solution by Soumava Chakraborty-Kolkata-India***

$$\frac{a^2}{\sqrt{a+3}} \overset{?}{\leq} \frac{15a^2+1}{32} \Leftrightarrow (a+3)(15a^2+1)^2 \overset{?}{\geq} 1024a^4$$

$$\Leftrightarrow (a-1)^2(225a^3+101a^2+7a+3) \overset{?}{\geq} 0 \to \text{true} \because a > 0$$

$$\therefore \frac{a^2}{\sqrt{a+3}} \leq \frac{15a^2+1}{32} \text{ and } \frac{b^2}{\sqrt{b+3}} \leq \frac{15b^2+1}{32} \text{ and so,}$$

$$\frac{a^2}{\sqrt{a+3}} + \frac{b^2}{\sqrt{b+3}} \leq \frac{15(a^2+b^2)+2}{32} \overset{a^2+b^2=2}{=} \frac{32}{32} = 1$$

$$\therefore \frac{a^2}{\sqrt{a+3}} + \frac{b^2}{\sqrt{b+3}} \leq 1 \ \forall\, a, b > 0 \,\middle|\, a^2 + b^2 = 2, '' ='' \text{ iff } a = b = 1 \text{ (QED)}$$

**2156. If $a, b, c > 0$ *then prove that* :**

$$\sum_{cyc} \frac{c^3 - a^3}{\sqrt{(1+a^2)(1+b^2)}} \geq 0$$

***Proposed by Nguyen Hung Cuong-Vietnam***

***Solution by Soumava Chakraborty-Kolkata-India***

$\boxed{\text{Case 1}}$ $a \geq \mathrm{b} \geq \mathrm{c}$ or $a \geq \mathrm{c} \geq \mathrm{b}$ or $\mathrm{b} \geq \mathrm{c} \geq a$ or $\mathrm{c} \geq \mathrm{b} \geq a$ *and then* :

$$\sum_{\text{cyc}} \frac{\mathrm{c}^3 - a^3}{\sqrt{(1+a^2)(1+\mathrm{b}^2)}} =$$

$$\frac{\mathrm{c}^3 - \mathrm{b}^3 + \mathrm{b}^3 - a^3}{\sqrt{(1+a^2)(1+\mathrm{b}^2)}} + \frac{a^3 - \mathrm{b}^3}{\sqrt{(1+\mathrm{b}^2)(1+\mathrm{c}^2)}} + \frac{\mathrm{b}^3 - \mathrm{c}^3}{\sqrt{(1+\mathrm{c}^2)(1+a^2)}}$$

$$= \frac{\mathrm{b}^3 - \mathrm{c}^3}{\sqrt{1+a^2}} \cdot \left(\frac{1}{\sqrt{1+\mathrm{c}^2}} - \frac{1}{\sqrt{1+\mathrm{b}^2}}\right) + \frac{a^3 - \mathrm{b}^3}{\sqrt{1+\mathrm{b}^2}} \cdot \left(\frac{1}{\sqrt{1+\mathrm{c}^2}} - \frac{1}{\sqrt{1+a^2}}\right)$$

$$= \frac{\mathrm{b}^3 - \mathrm{c}^3}{\sqrt{(1+a^2)(1+\mathrm{b}^2)(1+\mathrm{c}^2)}} \cdot \frac{1+\mathrm{b}^2 - 1 - \mathrm{c}^2}{\sqrt{1+\mathrm{b}^2} + \sqrt{1+\mathrm{c}^2}} +$$

$$\frac{a^3 - \mathrm{b}^3}{\sqrt{(1+a^2)(1+\mathrm{b}^2)(1+\mathrm{c}^2)}} \cdot \frac{1+a^2 - 1 - \mathrm{c}^2}{\sqrt{1+\mathrm{c}^2} + \sqrt{1+a^2}}$$

$$= \frac{(\mathrm{b}-\mathrm{c})^2(\mathrm{b}^2 + \mathrm{bc} + \mathrm{c}^2)(\mathrm{b}+\mathrm{c})}{\sqrt{(1+a^2)(1+\mathrm{b}^2)(1+\mathrm{c}^2)} . \sqrt{(1+\mathrm{b}^2)(1+\mathrm{c}^2)}} +$$

$$\frac{(a-\mathrm{b})(a-\mathrm{c})(a^2 + a\mathrm{b} + \mathrm{b}^2)(a+\mathrm{c})}{\sqrt{(1+a^2)(1+\mathrm{b}^2)(1+\mathrm{c}^2)} . \sqrt{(1+\mathrm{c}^2)(1+a^2)}} \geq 0$$

$(\because a \geq \mathrm{b} \geq \mathrm{c}$ or $a \geq \mathrm{c} \geq \mathrm{b}$ or $\mathrm{b} \geq \mathrm{c} \geq a$ or $\mathrm{c} \geq \mathrm{b} \geq a)$

$\boxed{\text{Case 2}}$ $\mathrm{b} \geq a \geq \mathrm{c}$ or $\mathrm{c} \geq a \geq \mathrm{b}$ *and then* : $\displaystyle\sum_{\text{cyc}} \frac{\mathrm{c}^3 - a^3}{\sqrt{(1+a^2)(1+\mathrm{b}^2)}} =$

$$\frac{\mathrm{c}^3 - a^3}{\sqrt{(1+a^2)(1+\mathrm{b}^2)}} + \frac{a^3 - \mathrm{c}^3 + \mathrm{c}^3 - \mathrm{b}^3}{\sqrt{(1+\mathrm{b}^2)(1+\mathrm{c}^2)}} + \frac{\mathrm{b}^3 - \mathrm{c}^3}{\sqrt{(1+\mathrm{c}^2)(1+a^2)}}$$

$$= \frac{\mathrm{c}^3 - a^3}{\sqrt{1+\mathrm{b}^2}} \cdot \left(\frac{1}{\sqrt{1+a^2}} - \frac{1}{\sqrt{1+\mathrm{c}^2}}\right) + \frac{\mathrm{b}^3 - \mathrm{c}^3}{\sqrt{1+\mathrm{c}^2}} \cdot \left(\frac{1}{\sqrt{1+a^2}} - \frac{1}{\sqrt{1+\mathrm{b}^2}}\right)$$

$$= \frac{\mathrm{c}^3 - a^3}{\sqrt{(1+a^2)(1+\mathrm{b}^2)(1+\mathrm{c}^2)}} \cdot \frac{1+\mathrm{c}^2 - 1 - a^2}{\sqrt{1+\mathrm{c}^2} + \sqrt{1+a^2}} +$$

$$\frac{\mathrm{b}^3 - \mathrm{c}^3}{\sqrt{(1+a^2)(1+\mathrm{b}^2)(1+\mathrm{c}^2)}} \cdot \frac{1+\mathrm{b}^2 - 1 - a^2}{\sqrt{1+\mathrm{b}^2} + \sqrt{1+a^2}}$$

$$= \frac{(\mathrm{c}-a)^2(\mathrm{c}^2 + \mathrm{c}a + a^2)(\mathrm{c}+a)}{\sqrt{(1+a^2)(1+\mathrm{b}^2)(1+\mathrm{c}^2)} . \sqrt{(1+\mathrm{c}^2)(1+a^2)}} +$$

$$\frac{(\mathrm{b}-\mathrm{c})(\mathrm{b}-a)(\mathrm{b}^2 + \mathrm{bc} + \mathrm{c}^2)(a+\mathrm{b})}{\sqrt{(1+a^2)(1+\mathrm{b}^2)(1+\mathrm{c}^2)} . \sqrt{(1+a^2)(1+\mathrm{b}^2)}} \geq 0 \ (\because \mathrm{b} \geq a \geq \mathrm{c} \text{ or } \mathrm{c} \geq a \geq \mathrm{b})$$

*and so*, combining both *cases*, $\displaystyle\sum_{\text{cyc}} \frac{\mathrm{c}^3 - a^3}{\sqrt{(1+a^2)(1+\mathrm{b}^2)}} \geq 0 \ \forall\, a, \mathrm{b}, \mathrm{c} > 0,$

$'' = ''$ iff $a = \mathrm{b} = \mathrm{c}$ (QED)

**2157. If $a, b, c \in (0, 1]$ then:**

$$\sqrt[3]{3^{\left(a^{\frac{1}{a}}+\frac{1}{b}-1\right)}}+\sqrt[3]{3^{\left(b^{\frac{1}{b}}+\frac{1}{c}-1\right)}}+\sqrt[3]{3^{\left(c^{\frac{1}{c}}+\frac{1}{a}-1\right)}}\ge\sqrt[3]{9+8\sqrt[3]{3^{\left(a^{\frac{1}{a}}+b^{\frac{1}{b}}+c^{\frac{1}{c}}+\frac{9}{a+b+c}\right)}}}$$

*Proposed by Pavlos Trifon-Greece*

*Solution by Tapas Das-India*

$$Let\ f(x)=x^{\frac{1}{x}}+\frac{1}{x},x\in(0,1]\ or\ f(u)=\left(\frac{1}{u}\right)^{u}+u=u^{-u}+u,$$

$$where\ u=\frac{1}{x}\ \&\ u\in[1,\infty)$$

$$f'(u)=1-\frac{1+\ln u}{u^{u}}=\frac{u^{u}-(1+\ln u)}{.u^{u}},clearly\ u^{u}>1\ as\ u\in[1,\infty)$$

$$Let\ h(u)=u^{u}-(1+\ln u)\ then\ h'(u)=u^{u}(1+\ln u)-\frac{1}{u}$$

$$u\ge 1\ then\ u^{u}(1+\ln u)>1\ and\frac{1}{u}<1,so\ h'(u)>1-1=0$$

$$so,h(u)increasing\ and\ h(u)>h(1)=0\ or,u^{u}-(1+\ln u)>0$$

$$so\ we\ can\ say\ f'(u)=\frac{u^{u}-(1+\ln u)}{.u^{u}}>0,$$

$$so\ f(u)increasing\ \&\ f(u)\ge f(1),f(1)=2$$

$$or,f(u)\ge 2\ \ or,u^{-u}+u\ge 2\ or,x^{\frac{1}{x}}+x\ge 2\ for$$

$$x\in(0,1]\ (1),\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\overset{Bergstrom}{\ge}\frac{9}{a+b+c}$$

$$Let\ 3^{\left(a^{\frac{1}{a}}+\frac{1}{b}-1\right)}=\frac{3^{\left(a^{\frac{1}{a}}+\frac{1}{b}\right)}}{3}=\frac{x}{3},\frac{3^{\left(b^{\frac{1}{b}}+\frac{1}{c}\right)}}{3}=\frac{y}{3},\frac{3^{\left(c^{\frac{1}{c}}+\frac{1}{a}\right)}}{3}=\frac{z}{3}$$

$$where\ x=3^{\left(a^{\frac{1}{a}}+\frac{1}{b}\right)},y=3^{\left(b^{\frac{1}{b}}+\frac{1}{c}\right)},z=3^{\left(c^{\frac{1}{c}}+\frac{1}{a}\right)}$$

$$\frac{x+y+z}{3}\overset{AM-GM}{\ge}\sqrt[3]{xyz}=\sqrt[3]{3^{\left(a^{\frac{1}{a}}+\frac{1}{b}\right)}3^{\left(b^{\frac{1}{b}}+\frac{1}{c}\right)}3^{\left(c^{\frac{1}{c}}+\frac{1}{a}\right)}}=\sqrt[3]{3^{\sum\left(a^{\frac{1}{a}}+\frac{1}{a}\right)}}\overset{(1)}{\ge}\sqrt[3]{3^{6}}=9\ (2)$$

$$R.H.S=\sqrt[3]{9+8\sqrt[3]{3^{\left(a^{\frac{1}{a}}+b^{\frac{1}{b}}+c^{\frac{1}{c}}+\frac{9}{a+b+c}\right)}}}\le\sqrt[3]{9+8\sqrt[3]{3^{\left(a^{\frac{1}{a}}+b^{\frac{1}{b}}+c^{\frac{1}{c}}+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}}}$$

$$=\sqrt[3]{9+8\sqrt[3]{xyz}}(A)$$

$$so\ L.H.S=\sqrt[3]{3^{\left(a^{\frac{1}{a}}+\frac{1}{b}-1\right)}}+\sqrt[3]{3^{\left(b^{\frac{1}{b}}+\frac{1}{c}-1\right)}}+\sqrt[3]{3^{\left(c^{\frac{1}{c}}+\frac{1}{a}-1\right)}}=\sqrt[3]{\frac{x}{3}}+\sqrt[3]{\frac{y}{3}}+\sqrt[3]{\frac{z}{3}}$$

$$now,(L.H.S)^{3}=\frac{x+y+z}{3}+3\prod\left(\sqrt[3]{\frac{x}{3}}+\sqrt[3]{\frac{y}{3}}\right)\overset{Cesaro}{\ge}\frac{x+y+z}{3}+3\times 8\sqrt[3]{\frac{xyz}{27}}$$

$$\overset{(2)}{\geq} 9+8\sqrt[3]{xyz} \ \ so\ L.H.S \geq \sqrt[3]{9+8\sqrt[3]{xyz}}\ (B)$$

$$From\ (A)\ \&\ (B) we\ get\ L.H.S \geq R.H.S$$

Equality holds for a=b=c=1.

**2158.** If $x_i > 0\ \ i \in \{1,2,3,\dots,n\},\ x_1+x_2+\dots+x_n = 1$ then prove:

$$\sum_{k=1}^{n} \frac{1}{\sqrt{1+x_k}} \geq n\sqrt{\frac{n}{n+1}}$$

*Proposed by Kostantinos Geronikolas-Greece*

*Solution by Mirsadix Muzefferov-Azerbaijan*

$$\sum_{k=1}^{n} \frac{1}{\sqrt{1+x_k}} = \sum_{k=1}^{n} \frac{1^3}{\sqrt{1+x_k}} \overset{C-B-S}{\geq} \frac{\left(\sum_1^n 1\right)^3}{n.\sum_{k=1}\sqrt{1+x_k}} \geq$$

$$= \frac{n^3}{n\sqrt{n(\sum 1 + \sum x_n)}} = \frac{n^2}{\sqrt{n(n+1)}} = \frac{n^{\frac{3}{2}}}{\sqrt{n+1}} = n\sqrt{\frac{n}{n+1}}$$

$$Equality\ holds\ when:\ x_1 = x_2 = x_3 = \dots = \frac{1}{n}$$

**2159.** If $x, y, z > 0, x+y+z = 1$ and $\lambda \leq \frac{9}{4}$ then:

$$\sum_{cyc} xy - \lambda xyz \leq \frac{9-\lambda}{27}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc} xy - \lambda xyz \overset{?}{\leq} \frac{9-\lambda}{27}$$

$$\Leftrightarrow \left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - \frac{1}{3}\left(\sum_{cyc} x\right)^3 + \frac{\lambda}{27}\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right) \overset{?}{\leq} 0 \ \left(\because \sum_{cyc} x = 1\right)$$

$$and \because \left(\sum_{cyc} x\right)^3 - 27xyz \overset{AM-GM}{\geq} 0\ and \because \lambda \leq \frac{9}{4} \therefore \text{it suffices to prove:}$$

$$\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - \frac{1}{3}\left(\sum_{cyc} x\right)^3 + \frac{1}{12}\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right) \overset{?}{\leq} 0$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)^3+9xyz\overset{?}{\ge}4\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\Leftrightarrow\sum_{cyc}x^3+3xyz\overset{?}{\ge}\sum_{cyc}x^2y+\sum_{cyc}xy^2$$

$$\to\text{true via Schur}\therefore\sum_{cyc}xy-\lambda xyz\le\frac{9-\lambda}{27}\ \forall\ x,y,z>0\,|\,x+y+z=1\text{ and }\lambda\le\frac{9}{4},$$

$$''=''\ \text{iff } x=y=z=\frac{1}{3}\ (\text{QED})$$

**2160.** *If $x,y,z>0$ $xy+yz+zx=3$ then:*

$$\sum_{cyc}\frac{x}{2x^4+y^2+z^2}\le\frac{3}{4xyz}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Mirsadix Muzefferov-Azerbaijan*

$$2x^4+y^2+z^2=x^4+x^4+y^2+z^2\overset{AM-GM}{\ge}$$

$$\ge4(x^4.x^4.y^2.z^2)^{\frac{1}{4}}=4x^2\sqrt{yz}$$

$$Analogously:\begin{cases}2y^4+x^2+z^2\ge4y^2\sqrt{xz}\\2z^4+y^2+x^2\ge4z^2\sqrt{yx}\end{cases}\ (*)$$

$$\sum_{cyc}\frac{x}{2x^4+y^2+z^2}\overset{(*)}{\le}\sum_{cyc}\frac{x}{4x^2\sqrt{yz}}=\sum_{cyc}\frac{1}{4x\sqrt{yz}}=$$

$$=\frac{\sqrt{xy}+\sqrt{yz}+\sqrt{zx}}{4xyz}\le\frac{\sqrt{3(xy+yz+zx)}}{4xyz}=\frac{\sqrt{3.3}}{4xyz}=\frac{3}{4xyz}$$

*Equality holds for $x=y=z=1$.*

**2161.** If $a,b>0$ and $a^3+b^3=2$ then prove that :

$$\frac{a^3}{\sqrt[3]{a+7}}+\frac{b^3}{\sqrt[3]{b+7}}\le1$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{a^3}{\sqrt[3]{a+7}}\overset{?}{\le}\frac{71a^3+1}{144}\Leftrightarrow(a+7)(71a^3+1)^3\overset{?}{\ge}(144)^3a^9$$

$$\Leftrightarrow(a-1)^2\left(\begin{matrix}23a^8+15a^7+7a^6+64a^6(5592a^2+3675a+1758)+\\4946a^5+3234a^4+1522a^3+23a^2+15a+7\end{matrix}\right)\overset{?}{\ge}0$$

$$\to\text{true}\because a>0\therefore\frac{a^3}{\sqrt[3]{a+7}}\le\frac{71a^3+1}{144}\text{ and }\frac{b^3}{\sqrt[3]{b+7}}\le\frac{71b^3+1}{144}\text{ and so,}$$

$$\frac{a^3}{\sqrt[3]{a+7}}+\frac{b^3}{\sqrt[3]{b+7}}\le\frac{71(a^3+b^3)+2}{144}\overset{a^3+b^3=2}{=}\frac{142+2}{144}=1$$

$$\therefore\frac{a^3}{\sqrt[3]{a+7}}+\frac{b^3}{\sqrt[3]{b+7}}\le 1\ \forall\, a,b>0\,|\,a^3+b^3=2,''=''\text{ iff } a=b=1\ (\text{QED})$$

**2162. If $a,b>0$ and $a+b=4$ then prove that :**

$$\frac{a^2}{\sqrt{a^3+8}}+\frac{b^2}{\sqrt{b^3+8}}\le 2$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{a^2}{\sqrt{a^3+8}}+\frac{b^2}{\sqrt{b^3+8}}=\sqrt{a}.\frac{a.\sqrt{a}}{\sqrt{a^3+8}}+\sqrt{b}.\frac{b.\sqrt{b}}{\sqrt{b^3+8}}\overset{CBS}{\le}$$

$$\sqrt{a+b}.\sqrt{\frac{a^3}{a^3+8}+\frac{b^3}{b^3+8}}\overset{a+b=4}{\le}2.\sqrt{\frac{a^3+8-8}{a^3+8}+\frac{b^3+8-8}{b^3+8}}$$

$$=2.\sqrt{2-8\left(\frac{1}{a^3+8}+\frac{2}{b^3+8}\right)}=2.\sqrt{2-\frac{8(a^3+b^3+16)}{a^3b^3+8(a^3+b^3)+64}}$$

$$\le 2.\sqrt{2-\frac{8(a^3+b^3+16)}{64+8(a^3+b^3)+64}}\ \left(\because 4=a+b\overset{AM-GM}{\ge}2.\sqrt{ab}\Rightarrow ab\le 4\right)$$

$$=2.\sqrt{2-\frac{8(a^3+b^3+16)}{8(a^3+b^3+16)}}=2\text{ and so, }\frac{a^2}{\sqrt{a^3+8}}+\frac{b^2}{\sqrt{b^3+8}}\le 2$$

$$\forall\, a,b>0\,|\,a+b=4,''=''\text{ iff } a=b=2\ (\text{QED})$$

**2163. If $a,b,c>0$ and $a+b+c\ge 3$ then prove that :**

$$\frac{1}{4a^4+b^4+c^4}+\frac{1}{4b^4+c^4+a^4}+\frac{1}{4c^4+a^4+b^4}\le\frac{1}{2}$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc}a^2\ge\frac{1}{3}\left(\sum_{cyc}a\right)^2\overset{a+b+c\ge 3}{\ge}3\Rightarrow 1\le\frac{1}{9}\left(\sum_{cyc}a^2\right)^2\Rightarrow\sum_{cyc}\frac{1}{4a^4+b^4+c^4}\le$$

$$\frac{1}{9}\left(\sum_{cyc}a^2\right)^2\cdot\left(\sum_{cyc}\frac{1}{4a^4+b^4+c^4}\right)=\frac{1}{9}\left(\sum_{cyc}x\right)^2\cdot\left(\sum_{cyc}\frac{1}{4x^2+y^2+z^2}\right)\begin{pmatrix}x=a^2,\\ y=b^2,\\ z=c^2\end{pmatrix}$$

$$=\frac{1}{9}\left(\sum_{cyc}x\right)^2\cdot\sum_{cyc}\frac{(4y^2+z^2+x^2)(4z^2+x^2+y^2)}{(4x^2+y^2+z^2)(4y^2+z^2+x^2)(4z^2+x^2+y^2)}\overset{?}{\leq}\frac{1}{2}$$

$$\Leftrightarrow 2\sum_{cyc}x^6+13\sum_{cyc}x^4y^2+13\sum_{cyc}x^2y^4+60x^2y^2z^2\overset{?}{\geq}4\sum_{cyc}x^5y+4\sum_{cyc}xy^5+$$

$$4xyz\sum_{cyc}x^3+12\sum_{cyc}x^3y^3+12xyz\left(\sum_{cyc}x^2y+\sum_{cyc}xy^2\right)\Leftrightarrow 2\sum_{cyc}x^6+$$

$$13\sum_{cyc}\left(x^2y^2\left(\sum_{cyc}x^2-z^2\right)\right)+60x^2y^2z^2\overset{?}{\geq}4\sum_{cyc}\left(xy\left(\sum_{cyc}x^4-z^4\right)\right)+$$

$$4xyz\sum_{cyc}x^3+12\sum_{cyc}x^3y^3+12xyz\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-3xyz\right)$$

$$\Leftrightarrow 2\sum_{cyc}x^6+13\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x^2y^2\right)+57x^2y^2z^2\boxed{\underset{(*)}{\overset{?}{\geq}}}4\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^4\right)+$$

$$12\sum_{cyc}x^3y^3+12xyz\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\ \&\ \text{assigning } y+z=A, z+x=B,$$

$x+y=C\Rightarrow A+B-C=2z>0, B+C-A=2x>0$ **and** $C+A-B=2y>0$
$\Rightarrow A+B>C, B+C>A, C+A>B\Rightarrow A,B,C$ *form sides of a* **triangle**
**with semiperimeter, circumradius and inradius** $=s,R,r$ (*say*) *and then* :

$$2\sum_{cyc}x=\sum_{cyc}A=2s\Rightarrow\sum_{cyc}x=s;\ \text{also } x=s-A, y=s-B, z=s-C\Rightarrow xyz=r^2s,$$

$$\sum_{cyc}xy=4Rr+r^2,\sum_{cyc}x^2=s^2-8Rr-2r^2,\sum_{cyc}x^2y^2=r^2((4R+r)^2-2s^2),\sum_{cyc}x^4$$

$$=(s^2-8Rr-2r^2)^2-2r^2((4R+r)^2-2s^2),\sum_{cyc}x^3y^3=(4Rr+r^2)^3-12Rr^3s^2$$

$$\text{and }\sum_{cyc}x^6=(s^3-12Rrs)^2-2((4Rr+r^2)^3-12Rr^3s^2)\ \&\ \text{via such substitutions}$$

*and following simplification*, $(*)$ **becomes** :

$$s^6-(32Rr+15r^2)s^4+r^2(376R^2+260Rr+55r^2)s^2-25r^3(4R+r)^3\boxed{\underset{(**)}{\overset{?}{\geq}}}0$$

$$\text{Since } (s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } (**), \text{it suffices to prove :}$$

$$\text{LHS of } (**) \overset{?}{\geq} (s^2-16Rr+5r^2)^3 \Leftrightarrow (8R-15r)s^4 - r(196R^2-370Rr+10r^2)s^2$$

$$+r^2(1248R^3-2520R^2r+450Rr^2-75r^3) \boxed{\underset{(***)}{\overset{?}{\geq}}} 0 \text{ and again, since}$$

$$(8R-15r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } (***),$$

$$\text{it suffices to prove : LHS of } (***) \overset{?}{\geq} (8R-15r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (6R^2-19Rr+14r^2)s^2 \boxed{\underset{(****)}{\overset{?}{\geq}}} r(80R^3-260R^2r+215Rr^2-30r^3) \text{ and}$$

$$\text{finally}, (6R^2-19Rr+14r^2)s^2 \overset{\text{Gerretsen}}{\geq} (6R^2-19Rr+14r^2)(16Rr-5r^2) \overset{?}{\geq}$$

$$r(80R^3-260R^2r+215Rr^2-30r^3) \Leftrightarrow 2r(R-2r)^2(8R-5r) \overset{?}{\geq} 0 \to \text{true}$$

$$\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true and so,}$$

$$\frac{1}{4a^4+b^4+c^4}+\frac{1}{4b^4+c^4+a^4}+\frac{1}{4c^4+a^4+b^4} \leq \frac{1}{2}\ \forall a,b,c>0 \,\middle|\, a+b+c\geq 3,$$

$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

**2164.** If $a,b,c>0, a+b+c=3$ then:

$$\sum_{cyc}\frac{a^2}{\sqrt{6a^2+13a+6}} \geq \frac{3}{5}$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Tapas Das-India*

*Lemma*:

$$\forall x>0, \sqrt{6x^2+13x+6} \leq \frac{5}{2}(x+1)$$

*Proof: we need to show*

$$\sqrt{6x^2+13x+6} \leq \frac{5}{2}(x+1)$$

$$25(x+1)^2 \geq 4(6x^2+13x+6) \text{ or } (x-1)^2 \geq 0 \text{ true}$$

$$\sum_{cyc}\frac{a^2}{\sqrt{6a^2+13a+6}} \overset{lemma}{\geq} \sum_{cyc}\frac{a^2}{\frac{5}{2}(a+1)} \overset{Bergstrom}{\geq} \frac{2}{5}\frac{(a+b+c)^2}{(a+b+c+3)} \overset{a+b+c=3}{=} \frac{3}{5}$$

Equality holds for a=b=c=1.

**2165.** If $\rho^2+\ln\rho=0,\ a,b,c>\rho$ then:

$$\sqrt[3]{abc.e^{a^2+b^2+c^2}}>1+\sqrt[3]{(a^2+\ln a)(b^2+\ln b)(c^2+\ln c)}$$

*Proposed by Pavlos Trifon-Greece*

*Solution by Tapas Das-India*

$$Let\ f(x)=\ x^2+\ln x\,,x>0\ then\ f'(x)=2x+\frac{1}{x}>0\ and\ f(\rho)=0\ (given)$$
$$so\ f\ is\ strictly\ increasing\ ,as\ a,b,c>\rho$$
$$therefore\ a^2+\ln a>0,b^2+\ln b>0\ ,c^2+\ln c>0$$

$$We\ need\ to\ show:$$
$$\sqrt[3]{abc.e^{a^2+b^2+c^2}}>1+\sqrt[3]{(a^2+\ln a)(b^2+\ln b)(c^2+\ln c)}\ \ (1)$$

$$Let\ x=\ a^2+\ln a\,,y=b^2+\ln b\,,z=c^2+\ln c$$

$$a^2+b^2+c^2=x+y+z-\ln a-\ln b-\ln c=x+y+z-\ln abc$$
$$so,abc\,.e^{a^2+b^2+c^2}=abc.\left(e^{(x+y+z-\ln abc)}\right)=e^{x+y+z}$$
$$hence\ inequality\ (1)\ becomes\ \sqrt[3]{e^{x+y+z}}>1+\sqrt[3]{xyz}$$

$$or\ e^{\frac{x+y+z}{3}}>1+\sqrt[3]{xyz}\ \ or\ e^{\sqrt[3]{xyz}}>1+\sqrt[3]{xyz}\ (AM-GM)$$
$$or\ \ e^t-(1+t)\overset{\sqrt[3]{xyz}=t>0}{>}0\ \ or\ 1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots-(1+t)>0$$
$$\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots>0\ True.$$

**2166.** If $x,y,z>0$ and $x^2+y^2+z^2\le 3$ then prove that :

$$\sqrt{\frac{x}{(2-y)(2-z)}}+\sqrt[3]{\frac{y}{(2-z)(2-x)}}+\sqrt[4]{\frac{z}{(2-x)(2-y)}}\le 3$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$x^2+y^2+z^2\le 3\Rightarrow x^2\le 3-y^2-z^2\ \text{and}$$
$$\because x^2>0\ \therefore 3-y^2-z^2\ge x^2>0\ \text{and so}$$
$$x\le\sqrt{3-y^2-z^2}=\sqrt{(3-y^2-z^2).1}\overset{\text{AM-GM}}{\le}\frac{3-y^2-z^2+1}{2}$$

$$\overset{?}{\leq}(2-y)(2-z)\Leftrightarrow 8-4(y+z)+2yz\overset{?}{\geq}4-y^2-z^2$$
$$\Leftrightarrow y^2+z^2+2yz-4(y+z)+4\overset{?}{\geq}0\Leftrightarrow (y+z)^2-4(y+z)+4\overset{?}{\geq}0$$
$$\Leftrightarrow (y+z-2)^2\overset{?}{\geq}0\to \text{true}\Rightarrow x\leq(2-y)(2-z)\Rightarrow\sqrt{\frac{x}{(2-y)(2-z)}}\overset{①}{\leq}1\ and$$
$$analogously, y\leq(2-z)(2-x)\Rightarrow\sqrt[3]{\frac{y}{(2-z)(2-x)}}\overset{②}{\leq}1\ and\ also,$$
$$z\leq(2-x)(2-y)\Rightarrow\sqrt[4]{\frac{z}{(2-x)(2-y)}}\overset{③}{\leq}1\ \therefore ①+②+③\Rightarrow$$
$$\sqrt{\frac{x}{(2-y)(2-z)}}+\sqrt[3]{\frac{y}{(2-z)(2-x)}}+\sqrt[4]{\frac{z}{(2-x)(2-y)}}\leq 3\ \forall\ x,y,z>0,$$
$$''='' \text{ iff } y+z=2\wedge 3-y^2-z^2=1\wedge x^2=3-y^2-z^2$$
$$\Rightarrow ''='' \text{ iff } y+z=2\wedge yz=1\wedge x=1\Rightarrow ''='' \text{ iff } x=y=z=1\ (\text{QED})$$

**2167.** If $a,b,c>0, a+b+c\leq 3$ then:

$$\frac{a+b}{c^2}+\frac{b+c}{a^2}+\frac{a+c}{b^2}\geq 6$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Mirsadix Muzefferov-Azerbaijan*

$$\frac{a+b}{c^2}+\frac{b+c}{a^2}+\frac{a+c}{b^2}\overset{AM-GM}{\geq}\frac{2\sqrt{ab}}{c^2}+\frac{2\sqrt{bc}}{a^2}+\frac{2\sqrt{ac}}{b^2}=$$
$$2\left(\frac{\sqrt{ab}}{c^2}+\frac{\sqrt{bc}}{a^2}+\frac{\sqrt{ac}}{b^2}\right)\overset{AM-GM}{\geq}6\left(\frac{\sqrt{(abc)^2}}{a^2b^2c^2}\right)^{\frac{1}{3}}=\frac{6}{\sqrt[3]{abc}}\geq\frac{6}{1}=6$$
$$\therefore\left(\sqrt[3]{abc}\leq\frac{a+b+c}{3}\leq\frac{3}{3}=1\right)$$

$$Equality\ \ holds\ for\ a=b=c=1$$

**2168.** If $a,b>0$ and $a+b=2ab$ then:

$$\frac{a}{b^2}+\frac{b}{a^2}\ge 2$$

*Proposed by Nguyen Hung Cuong – Vietnam*

*Solution by Daniel Sitaru – Romania*

$$\frac{a}{b^2}+\frac{b}{a^2}\ge 2 \Leftrightarrow a^3+b^3\ge 2a^2b^2$$

$$(a+b)(a^2-ab+b^2)\ge 2a^2b^2,\ \ 2ab(a^2-ab+b^2)\ge 2a^2b^2$$

$$a^2-ab+b^2\ge ab,\qquad a^2-2ab+b^2\ge 0$$

$$(a-b)^2\ge 0$$

Equality holds for $a=b$.

**2169.**

**If $a,b,c\ge 0$ and $a+b+c=3$ then prove that :**

$$\frac{5ab}{3c+ab}+\frac{3bc}{3a+bc}+\frac{3ca}{3b+ca}\le 5$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$3c+ab\overset{a+b+c=3}{=}\frac{3c+a(3-c-a)}{2}+\frac{3c+b(3-b-c)}{2}$$

$$=\frac{(3-a)(c+a)}{2}+\frac{(3-b)(b+c)}{2}\overset{a+b+c=3}{=}\frac{(b+c)(c+a)}{2}+\frac{(c+a)(b+c)}{2}$$

$$\Rightarrow 3c+ab\ \boxed{\overset{(1)}{=}}\ (b+c)(c+a)$$

$$\text{Also, } 3a+bc\overset{a+b+c=3}{=}3a+c(3-c-a)=(3-c)(c+a)\overset{a+b+c=3}{=}(a+b)(c+a)$$

$$\Rightarrow 3a+bc\ \boxed{\overset{(2)}{=}}\ (a+b)(c+a)$$

$$\text{And, } 3b+ca\overset{a+b+c=3}{=}3b+c(3-b-c)=(3-c)(b+c)\overset{a+b+c=3}{=}(a+b)(b+c)$$

$$\Rightarrow 3b+ca\ \boxed{\overset{(3)}{=}}\ (a+b)(b+c)$$

$$\text{Now, }\frac{5ab}{3c+ab}+\frac{3bc}{3a+bc}+\frac{3ca}{3b+ca}\overset{?}{\le}5\Leftrightarrow\frac{3bc}{3a+bc}+\frac{3ca}{3b+ca}\overset{?}{\le}5-\frac{5ab}{3c+ab}$$

$$=\frac{15c}{3c+ab}\Leftrightarrow\frac{5}{3c+ab}\overset{?}{\ge}\frac{a}{3b+ca}+\frac{b}{3a+bc}\ (\because c\ge 0)$$

$$\overset{\text{via (1),(2) and (3)}}{\Leftrightarrow}\frac{5}{(b+c)(c+a)}\overset{?}{\ge}\frac{a}{(a+b)(b+c)}+\frac{b}{(a+b)(c+a)}$$

$$\Leftrightarrow 5(a+b) \overset{?}{\geq} a(c+a)+b(b+c)=c(a+b)+a^2+b^2$$

$$\Leftrightarrow (3-c+2)(a+b) \overset{?}{\geq} a^2+b^2 \overset{a+b+c=3}{\Leftrightarrow} (a+b+2)(a+b) \overset{?}{\geq} a^2+b^2$$

$\Leftrightarrow 2ab+2(a+b) \overset{?}{\geq} 0 \rightarrow$ true $\because a,b \geq 0$ and this last inequality is a strict one since $a,b$ cannot be simultaneously equal to zero as that would make $\frac{3bc}{3a+bc}$ and $\frac{3ca}{3b+ca}$ undefined $\therefore \frac{5ab}{3c+ab}+\frac{3bc}{3a+bc}+\frac{3ca}{3b+ca} \leq 5$

$$\forall\, a,b,c \geq 0 \,|\, a+b+c=3, '' ='' \text{ iff } \begin{pmatrix} c=0 \\ a=k \\ b=3-k \\ \text{with } 0<k<3 \end{pmatrix} \text{(QED)}$$

**2170.** If $x,y,z>0, x^3+y^3+z^3=3$ then :

$$\sum_{cyc}\left(\frac{x^2+1}{x+1}\right)^3 \geq 3$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\left(\frac{x^2+1}{x+1}\right)^3 \overset{?}{\geq} \frac{x^3+1}{2} \Leftrightarrow 2(x^2+1)^3 \overset{?}{\geq} (x^3+1)(x+1)^3$$

$$\Leftrightarrow (x^2+x+1)(x-1)^4 \overset{?}{\geq} 0 \rightarrow \text{true} \therefore \left(\frac{x^2+1}{x+1}\right)^3 \geq \frac{x^3+1}{2} \text{ and analogs}$$

$$\Rightarrow \sum_{cyc}\left(\frac{x^2+1}{x+1}\right)^3 \geq \sum_{cyc}\frac{x^3+1}{2} = \frac{1}{2}(3)+\frac{3}{2}\left(\because \sum_{cyc}x^3=3\right)=3$$

$$\text{So, } \sum_{cyc}\left(\frac{x^2+1}{x+1}\right)^3 \geq 3\ \forall\, x,y,z>0 \,|\, x^3+y^3+z^3=3, '' ='' \ x=y=z=1 \text{ (QED)}$$

**2171.** If $x,y,z>0$ and $x^2+y^2+z^2+xyz=4$ then prove that :

$$\sqrt{2-x}+\sqrt{2-y}+\sqrt{2-z} \geq 2+\sqrt{(2-x)(2-y)(2-z)}$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Firstly, } (4-y^2),(4-z^2)=(x^2+z^2+xyz),(x^2+y^2+xyz)$$

$$\overset{x,y,z>0}{>} 0 \text{ and we have}: x^2+y^2+z^2+xyz=4 \Rightarrow x^2+x.yz+(y^2+z^2-4)=0$$

$$\Rightarrow x=\frac{-yz\pm\sqrt{y^2z^2-4(y^2+z^2-4)}}{2}=\frac{-yz\pm\sqrt{y^2(z^2-4)-4(z^2-4)}}{2}$$

$$=\frac{-yz\pm\sqrt{(4-y^2)(4-z^2)}}{2}=\frac{-yz+\sqrt{(4-y^2)(4-z^2)}}{2}\ (\because x>0)$$

$$\Rightarrow 2-x=2-\frac{-yz+\sqrt{(4-y^2)(4-z^2)}}{2}=\frac{8+2yz-2\sqrt{(4-y^2)(4-z^2)}}{4}$$

$$\overset{A-G}{\geq}\frac{8+2yz-(8-y^2-z^2)}{4}=\frac{y^2+z^2+2yz}{4}=\frac{(y+z)^2}{4}$$

$$\Rightarrow \sqrt{2-x}\geq\frac{y+z}{2} \text{ and analogs} \Rightarrow \sum_{zyz}\sqrt{2-x}\ \boxed{\overset{①}{\geq}}\ \sum_{zyz}\frac{y+z}{2}=\sum_{cyc}x$$

$$\Rightarrow \sum_{cyc}x\leq\sum_{cyc}\sqrt{2-x}\overset{CBS}{\leq}\sqrt{3}.\sqrt{6-\sum_{cyc}x}\Rightarrow t^2\leq 18-3t\left(t=\sum_{cyc}x\right)$$

$$\Rightarrow (t-3)(t+6)\leq 0\Rightarrow t=\sum_{cyc}x\ \boxed{\overset{②}{\leq}}\ 3 \text{ and we have}: \left(\sum_{cyc}x\right)^2-4=$$

$$\sum_{cyc}x^2+2\sum_{cyc}xy-\left(\sum_{cyc}x^2+xyz\right)=2xyz\sum_{cyc}\frac{1}{x}-xyz\overset{Bergstrom}{\geq}\frac{18xyz}{\sum_{cyc}x}-xyz$$

$$\overset{via\ ②}{\geq} 6xyz-xyz=5xyz>0\Rightarrow\left(\sum_{cyc}x\right)^2>4\Rightarrow\sum_{cyc}x\ \boxed{\overset{③}{\geq}}\ 2$$

$$\therefore \boxed{\sum_{zyz}\sqrt{2-x}-2-\sqrt{(2-x)(2-y)(2-z)}}\overset{via\ ①}{\geq}$$

$$\sum_{cyc}x-2-\sqrt{8+2\sum_{cyc}xy-4\sum_{cyc}x+\sum_{cyc}x^2-4}\left(\because -xyz=\sum_{cyc}x^2-4\right)$$

$$=\sum_{cyc}x-2-\sqrt{4-4\sum_{cyc}x+\left(\sum_{cyc}x\right)^2}=\sum_{cyc}x-2-\sqrt{\left(\sum_{cyc}x-2\right)^2}$$

$$\overset{via\ ③}{=}\sum_{cyc}x-2-\left(\sum_{cyc}x-2\right)=0 \text{ and so,}$$

$$\sqrt{2-x}+\sqrt{2-y}+\sqrt{2-z}\geq 2+\sqrt{(2-x)(2-y)(2-z)}$$

$$\forall\, x,y,z>0\,\middle|\,x^2+y^2+z^2+xyz=4,''=''\text{ iff } x=y=z=1\ (QED)$$

**2172.** Let $a_1, a_2, a_3, a_4,$ be positive real numbers $a_1.a_2.a_3.a_4 = 1$. Prove that :

$$\sum_{cyc} \frac{1}{a_1^3(a_2+a_3+a_4)} \geq \frac{4}{3}$$

*Proposed by Gheorghe Crăciun-Romania*

*Solution by Khaled Abd Imouti-Syria*

$$LHS = \frac{1}{a_1^3(a_2+a_3+a_4)} + \frac{1}{a_2^3(a_1+a_3+a_4)} + \frac{1}{a_3^3(a_1+a_2+a_4)} + \frac{1}{a_4^3(a_1+a_2+a_3)}$$

$$LHS = \frac{\frac{1}{a_1^2}}{a_1(a_2+a_3+a_4)} + \frac{\frac{1}{a_2^2}}{a_2(a_1+a_3+a_4)} + \frac{\frac{1}{a_3^2}}{a_3(a_1+a_2+a_4)} + \frac{\frac{1}{a_4^2}}{a_4(a_1+a_2+a_3)}$$

$$LHS = \frac{(\frac{1}{a_1})^2}{a_1(a_2+a_3+a_4)} + \frac{(\frac{1}{a_2})^2}{a_2(a_1+a_3+a_4)} + \frac{(\frac{1}{a_3})^2}{a_3(a_1+a_2+a_4)} + \frac{(\frac{1}{a_4})^2}{a_4(a_1+a_2+a_3)}$$

Suppose : $x = \frac{1}{a_1}$ , $y = \frac{1}{a_2}$ , $z = \frac{1}{a_3}$ , $t = \frac{1}{a_4}$

$$LHS = \frac{x^2}{\frac{1}{xy}+\frac{1}{xz}+\frac{1}{xt}} + \frac{y^2}{\frac{1}{yx}+\frac{1}{yz}+\frac{1}{yt}} + \frac{z^2}{\frac{1}{zx}+\frac{1}{zy}+\frac{1}{zt}} + \frac{t^2}{\frac{1}{tx}+\frac{1}{ty}+\frac{1}{tz}}$$

$$LHS = \frac{x^3yzt}{zt+yt+zy} + \frac{y^3xzt}{zt+xt+xz} + \frac{z^3xyt}{ty+xt+xy} + \frac{t^3xyz}{zy+zx+xy}$$

$$LHS = \frac{x^2}{zt+yt+zy} + \frac{y^2}{zt+xt+xz} + \frac{z^2}{ty+xt+xy} + \frac{t^2}{zy+zx+xy}$$

$$LHS \geq \frac{(x+y+z+t)^2}{2(xy+xz+xt+yz+yt+zt)} \quad by\ Bergstrom's\ inequality$$

But : $(x+y+z+t)^2 \geq \frac{2}{3}(xy+xz+xt+yz+yt+zt) \Rightarrow So: L \geq \frac{\frac{2}{3}}{2} = \frac{4}{3}$

**2173.** If $a, b, c > 0, abc = 1$ then :

$$\sum_{cyc} \sqrt[4]{\frac{2a}{a+1}} \leq 3$$

*Proposed by Marin Chirciu-Romania*

***Solution by Soumava Chakraborty-Kolkata-India***

$$\text{Since } abc = 1, \text{we can assign}: a = \frac{yz}{x^2}, b = \frac{zx}{y^2}, c = \frac{xy}{z^2} \text{ and then}:$$

$$\sum_{cyc} \sqrt[4]{\frac{2a}{a+1}} \overset{CBS}{\leq} \sqrt{3}.\sqrt{\sum_{cyc} \sqrt{\frac{2yz}{yz+x^2}}} \overset{?}{\leq} 3$$

$$\Leftrightarrow \sum_{cyc} \sqrt{yz(zx+y^2)(xy+z^2)} \underset{(*)}{\overset{?}{\leq}} \frac{3}{\sqrt{2}}.\sqrt{(yz+x^2)(zx+y^2)(xy+z^2)}$$

$$\text{Now,} \sum_{cyc} \sqrt{yz(zx+y^2)(xy+z^2)} \overset{CBS}{\leq} \sqrt{\sum_{cyc} y(xy+z^2)}.\sqrt{\sum_{cyc} z(zx+y^2)}$$

$$= 2\sqrt{\left(\sum_{cyc} xy^2\right)\left(\sum_{cyc} x^2y\right)} \overset{?}{\leq} \frac{3}{\sqrt{2}}.\sqrt{(yz+x^2)(zx+y^2)(xy+z^2)}$$

$$\Leftrightarrow 9\left(xyz\sum_{cyc} x^3 + \sum_{cyc} x^3y^3 + 2x^2y^2z^2\right) \overset{?}{\geq} 8\left(xyz\sum_{cyc} x^3 + \sum_{cyc} x^3y^3 + 3x^2y^2z^2\right)$$

$$\Leftrightarrow xyz\sum_{cyc} x^3 + \sum_{cyc} x^3y^3 \overset{?}{\geq} 6x^2y^2z^2 \rightarrow \text{true} \because xyz\sum_{cyc} x^3 + \sum_{cyc} x^3y^3 \overset{AM-GM}{\geq}$$

$$3x^2y^2z^2 + 3x^2y^2z^2 = 6x^2y^2z^2 \Rightarrow (*) \text{ is true} \therefore \sum_{cyc} \sqrt[4]{\frac{2a}{a+1}} \leq 3\ \forall\ abc = 1,$$

$$'' ='' \quad a = b = c = 1 \text{ (QED)}$$

**2174.** If $x, y, z > 0, \sum_{cyc} \frac{1}{x} = 3$ then :

$$\sum_{cyc} \frac{1}{x^3+x+2} \leq \frac{3}{4}$$

***Proposed by Marin Chirciu-Romania***

***Solution by Soumava Chakraborty-Kolkata-India***

$$\frac{1}{x^3+x+2} \overset{?}{\leq} \frac{1}{4x} \Leftrightarrow x^3 - 3x + 2 \overset{?}{\geq} 0 \Leftrightarrow (x-1)^2(x+2) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because x > 0 \therefore \frac{1}{x^3+x+2} \leq \frac{1}{4x} \text{ and analogs} \therefore \sum_{cyc} \frac{1}{x^3+x+2} \leq \frac{1}{4}\left(\sum_{cyc} \frac{1}{x}\right)$$

$$=\frac{3}{4}\ \forall\, x,y,z>0 \,\Big|\, \sum_{cyc}\frac{1}{x}=3,\ ''=''\ \text{ iff } x=y=z=1\ (QED)$$

**2175. If $a,b,c\geq 0$ and $a+b+c=2$ then prove that :**

$$a^2+b^2+c^2\geq 2(a^3b^3+b^3c^3+c^3a^3+4a^2b^2c^2)$$

***Proposed by Nguyen Hung Cuong-Vietnam***

***Solution by Soumava Chakraborty-Kolkata-India***

When exactly 2 variables equal to zero and WLOG we may assume $b=c=0\ (a=2)$, then : $LHS=4>0=RHS$

When exactly 1 variable equals to zero and WLOG we may assume $a=0$

$(b+c=2)$, then : inequality becomes : $b^2+c^2\overset{?}{\geq}2b^3c^3$ and $\because b^2+c^2\overset{A-G}{\geq}2bc$

$\therefore$ it suffices to prove : $2bc\overset{?}{\geq}2b^3c^3\Leftrightarrow bc\overset{?}{\leq}1\to$ true $\because 2=b+c\overset{A-G}{\geq}2\sqrt{bc}$ and so, we now focus on the case when : $a,b,c>0$ and assigning $b+c=x$, $c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0\ \&\ z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); then :

$$\sum_{cyc}a=s,\ abc=r^2s,\ \sum_{cyc}ab=4Rr+r^2,\ \sum_{cyc}a^2=s^2-8Rr-2r^2,$$

$$\sum_{cyc}a^3b^3=(4Rr+r^2)^3-12Rr^3s^2 \text{ and then :}$$

$$\sum_{cyc}a^2\overset{?}{\geq}2\left(\sum_{cyc}a^3b^3+4a^2b^2c^2\right)\overset{a+b+c=2}{\Leftrightarrow}$$

$$\frac{1}{16}\cdot\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a\right)^4\overset{?}{\geq}2\left(\sum_{cyc}a^3b^3+4a^2b^2c^2\right)$$

$$\Leftrightarrow (s^2-8Rr-2r^2)s^4\overset{?}{\geq}32((4Rr+r^2)^3-12Rr^3s^2+4r^4s^2)$$

$$\Leftrightarrow s^6-(8Rr+2r^2)s^4+r^3(384R-128r)s^2-r^3(4R+r)^3\ \boxed{\underset{(*)}{\overset{?}{\geq}}}\ 0$$

and $\because (s^2-16Rr+5r^2)^3\overset{Gerretsen}{\geq}0\ \therefore$ in order to prove $(*)$, it suffices to prove :

LHS of $(*)\overset{?}{\geq}(s^2-16Rr+5r^2)^3\Leftrightarrow (40R-17r)s^4-$

$$r(768R^2-864Rr+203r^2)s^2+r^2(2048R^3-5376R^2r+816Rr^2-157r^3)\ \boxed{\underset{(**)}{\overset{?}{\geq}}}\ 0$$

and $\because (40R-17r)(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\geq}0\ \therefore$ in order to prove $(**)$,

$$\text{it suffices to prove : LHS of } (**) \overset{?}{\geq} (40R - 17r)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (512R^2 - 80Rr - 33r^2)s^2 \boxed{\underset{(***)}{\overset{?}{\geq}}} r(8192R^3 - 5376R^2r + 2904Rr^2 - 268r^3)$$

$$\text{and finally, LHS of } (***) \overset{\text{Gerretsen}}{\geq} (512R^2 - 80Rr - 33r^2)(16Rr - 5r^2) \overset{?}{\geq}$$

$$\text{RHS of } (***) \Leftrightarrow r^2\left((1536R + 40r)(R - 2r) + 513r^2\right) \overset{?}{\geq} 0 \to \text{true}$$

$$(\text{strict inequality}) \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \forall \Delta\, XYZ_{s,R,r}$$

$$\therefore \sum_{cyc} a^2 > 2\left(\sum_{cyc} a^3b^3 + 4a^2b^2c^2\right) \text{ whenever } a,b,c > 0 \text{ and combining all cases,}$$

$$\text{we have : } \sum_{cyc} a^2 \geq 2\left(\sum_{cyc} a^3b^3 + 4a^2b^2c^2\right) \forall\, a,b,c \geq 0 \,|\, a+b+c = 2,$$

$$'' ='' \text{ iff } \begin{pmatrix} a = 0 \\ b = c = 1 \end{pmatrix} \text{ or } \begin{pmatrix} b = 0 \\ c = a = 1 \end{pmatrix} \text{ or } \begin{pmatrix} c = 0 \\ a = b = 1 \end{pmatrix} \text{ (QED)}$$

**2176. If $a,b,c \geq 0$ and $ab + bc + ca > 0$ then prove that :**

$$\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + a^2} \geq \frac{10}{(a + b + c)^2}$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

When exactly 1 variable equals to zero and WLOG we may assume $a = 0$

$$(b,c > 0), \text{then, inequality becomes : } \frac{1}{b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2} \overset{?}{\geq} \frac{10}{(b + c)^2}$$

$$\Leftrightarrow b^6 + 2b^5c - 6b^4c^2 + 6b^3c^3 - 6b^2c^4 + 2bc^5 + c^6 \overset{?}{\geq} 0$$

$\Leftrightarrow (b - c)^2(b^4 + 4b^3c + b^2c^2 + 4bc^3 + c^4) \overset{?}{\geq} 0 \to$ true and we now focus on the case when : $a,b,c > 0$ and assigning $b + c = x, c + a = y, a + b = z \Rightarrow$ $x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z,$ $y + z > x, z + x > y \Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s,R,r$ (say); then : $\sum_{cyc} a = s, abc = r^2s,$

$$\sum_{cyc} ab = 4Rr + r^2, \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2, \sum_{cyc} a^2b^2 = r^2((4R + r)^2 - 2s^2)$$

$$\text{and then : } \sum_{cyc} \frac{1}{b^2 + c^2} \overset{?}{>} \frac{10}{(a + b + c)^2}$$

$$\Leftrightarrow \left(\left(\sum_{cyc} a^2\right)^2 + \sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right)^2 \overset{?}{>} 10\left(\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) - a^2b^2c^2\right)$$

$$\Leftrightarrow s^2\left((s^2-8Rr-2r^2)^2 + r^2((4R+r)^2-2s^2)\right) \overset{?}{>}$$
$$10\left((s^2-8Rr-2r^2)(r^2)((4R+r)^2-2s^2) - r^4s^2\right)$$

$$\Leftrightarrow s^6 - (16Rr-14r^2)s^4 - r^2(80R^2+200Rr+35r^2)s^2 + 20r^3(4R+r)^3 \boxed{\underbrace{\overset{?}{>}}_{(*)}} 0$$

$and \because (s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(*)$, it suffices to prove :

$$\text{LHS of } (*) \overset{?}{>} (s^2-16Rr+5r^2)^3 \Leftrightarrow (32R-r)s^4 -$$
$$r(848R^2-280Rr+110r^2)s^2 + r^2(5376R^3-2880R^2r+1440Rr^2-105r^3) \boxed{\underbrace{\overset{?}{>}}_{(**)}} 0$$

$and \because (32R-r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore$ in order to prove $(**)$,

it suffices to prove : LHS of $(**) \overset{?}{>} (32R-r)(s^2-16Rr+5r^2)^2$

$$\Leftrightarrow (44R^2-18Rr-25r^2)s^2 \boxed{\underbrace{\overset{?}{>}}_{(***)}} r(704R^3-624R^2r-120Rr^2+20r^3) \; and$$

$finally$, LHS of $(***) \overset{\text{Gerretsen}}{\geq} (44R^2-18Rr-25r^2)(16Rr-5r^2) \overset{?}{>}$ RHS of $(***)$

$\Leftrightarrow r^2(116R^2-190Rr+105r^2) \overset{?}{>} 0 \to$ true (strict inequality) $\because R \overset{\text{Euler}}{\geq} 2r$

$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*)$ is true $\forall \Delta XYZ_{s,R,r} \therefore \sum_{cyc} \frac{1}{b^2+c^2} > \frac{10}{(a+b+c)^2}$

whenever $a, b, c > 0$ and combining $all$ cases, we have :

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \geq \frac{10}{(a+b+c)^2} \; \forall a,b,c \geq 0 \,|\, ab+bc+ca > 0,$$

$$'' ='' \text{ iff } \begin{pmatrix} a=0 \\ b=c>0 \end{pmatrix} \text{ or } \begin{pmatrix} b=0 \\ c=a>0 \end{pmatrix} \text{ or } \begin{pmatrix} c=0 \\ a=b>0 \end{pmatrix} \text{ (QED)}$$

**2177.** If $x, y, z > 0$ $and$ $x+y+z=3$ then prove that :

$$\frac{1}{1+xy} + \frac{1}{1+yz} + \frac{1}{1+zx} \geq \frac{6}{\sqrt{x}+\sqrt{y}+\sqrt{z}} - \frac{1}{2}$$

***Proposed by Nguyen Hung Cuong-Vietnam***

***Solution by Soumava Chakraborty-Kolkata-India***

$$\sum_{cyc} \frac{1}{1+yz} = \sum_{cyc} \frac{1+yz-yz}{1+yz} = 3 - \sum_{cyc} \frac{yz}{1+yz} \overset{\text{AM-GM}}{\geq} 3 - \sum_{cyc} \frac{yz}{2\sqrt{yz}}$$

$$= 3 - \frac{1}{2}\sum_{cyc}\sqrt{yz} \overset{?}{\geq} \frac{6}{\sum_{cyc}\sqrt{x}} - \frac{1}{2} \Leftrightarrow \frac{7 - \sum_{cyc} bc}{2} \overset{?}{\geq} \frac{6}{\sum_{cyc} a}\ (a = \sqrt{x}, b = \sqrt{y}, c = \sqrt{z})$$

$$\Leftrightarrow \frac{\frac{7}{3}\left(\sum_{cyc} a^2\right) - \sum_{cyc} ab}{2} \overset{?}{\geq} \frac{2\left(\sum_{cyc} a^2\right)}{\sum_{cyc} a}\cdot\sqrt{\frac{\sum_{cyc} a^2}{3}}\ \left(\because \sum_{cyc} x = 3 \Leftrightarrow \sum_{cyc} a^2 = 3\right)$$

$$\Leftrightarrow \left(7\sum_{cyc} a^2 - 3\sum_{cyc} ab\right)^2\left(\sum_{cyc} a\right)^2 \boxed{\overset{?}{\underset{(\bullet)}{\geq}}} 48\left(\sum_{cyc} a^2\right)^3 \left(\begin{matrix}\because \frac{7}{3}\left(\sum_{cyc} a^2\right) \geq \\ \frac{7}{3}\left(\sum_{cyc} ab\right) > \sum_{cyc} ab\end{matrix}\right)$$

Assigning $b + c = X, c + a = Y, a + b = Z \Rightarrow X + Y - Z = 2c > 0, Y + Z - X = 2a > 0$ and $Z + X - Y = 2b > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ *form* sides of $a$ triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say); and then : $a = s - X, b = s - Y, c = s - Z \therefore \sum_{cyc} a = s, \sum_{cyc} ab = 4Rr + r^2$,

$$\sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \text{ and hence, } (\bullet) \Leftrightarrow s^6 + (200Rr + 50r^2)s^4 -$$

$$r^2(4592R^2 + 2296Rr + 287r^2)s^2 + 384r^3(4R + r)^3 \boxed{\overset{?}{\underset{(*)}{\geq}}} 0 \text{ and } \because$$

$$(s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \overset{?}{\geq} (s^2 - 16Rr + 5r^2)^3 \Leftrightarrow (248R + 35r)s^4 -$$

$$r(5360R^2 + 1816Rr + 362r^2)s^2 + r^2(28672R^3 + 14592R^2r + 5808Rr^2 + 259r^3)$$

$$\boxed{\overset{?}{\underset{(**)}{\geq}}} 0 \text{ and } \because (248R + 35r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**) \overset{?}{\geq} (248R + 35r)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (322R^2 - 397Rr - 89r^2)s^2 \boxed{\overset{?}{\underset{(***)}{\geq}}} r(4352R^3 - 5664R^2r - 651Rr^2 + 77r^3)$$

$$\& (322R^2 - 397Rr - 89r^2)s^2 \overset{\text{Gerretsen}}{\geq} (322R^2 - 397Rr - 89r^2)(16Rr - 5r^2)$$

$$\overset{?}{\geq} \mathrm{r}(4352\mathrm{R}^3 - 5664\mathrm{R}^2\mathrm{r} - 651\mathrm{Rr}^2 + 77\mathrm{r}^3) \Leftrightarrow 400\mathrm{t}^3 - 1149\mathrm{t}^2 + 606\mathrm{t} + 184 \overset{?}{\geq} 0$$

$$\left(\mathrm{t} = \frac{\mathrm{R}}{\mathrm{r}}\right) \Leftrightarrow (\mathrm{t}-2)\big((\mathrm{t}-2)(400\mathrm{t}+451)+810\big) \overset{?}{\geq} 0 \rightarrow \text{true} \because \mathrm{t} \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \forall \Delta\, \mathrm{XYZ}_{\mathrm{s,R,r}} \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \frac{1}{1+xy} + \frac{1}{1+yz} + \frac{1}{1+zx} \geq \frac{6}{\sqrt{x}+\sqrt{y}+\sqrt{z}} - \frac{1}{2}$$

$$\forall x,y,z > 0 \,|\, x+y+z = 3, ''='' \text{ iff } x = y = z = 1 \text{ (QED)}$$

**2178.** If $x, y, z > 0$ and $xyz = 1$ then prove that :

$$\sum_{cyc} x^2 + 6 \geq \frac{3}{2}\left(\sum_{cyc} x + \sum_{cyc} \frac{1}{x}\right)$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Let } \sqrt[2]{x} = a, \sqrt[3]{y} = b, \sqrt[3]{z} = c \text{ and then : } \sum_{cyc} x^2 + 6 \overset{?}{\geq} \frac{3}{2}\left(\sum_{cyc} x + \sum_{cyc} \frac{1}{x}\right)$$

$$\Leftrightarrow 2\left(\sum_{cyc} a^6 + 6a^2b^2c^2\right) \boxed{\overset{?}{\underset{(\bullet)}{\geq}}} 3\left(abc\sum_{cyc} a^3 + \sum_{cyc} a^3b^3\right) \; (\because xyz = 1 \Rightarrow abc = 1) \;\&$$

assigning $b + c = X, c + a = Y, a + b = Z \Rightarrow X + Y - Z = 2c > 0, Y + Z - X = 2a > 0$ and $Z + X - Y = 2b > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ form sides of a triangle XYZ with semiperimeter, circumradius, inradius $= s, R, r$ (say);

$$\text{and then : } a = s - X, b = s - Y, c = s - Z \therefore abc = r^2s, \sum_{cyc} a = s,$$

$$\sum_{cyc} ab = 4Rr + r^2, \sum_{cyc} a^3 = s^3 - 12Rrs \;\&\; \text{so}, \sum_{cyc} a^3b^3 = (4Rr + r^2)^3 - 12Rr^3s^2$$

$$\text{and } \sum_{cyc} a^6 = (s^3 - 12Rrs)^2 - 2\big((4Rr + r^2)^3 - 12Rr^3s^2\big) \text{ and hence, } (\bullet) \Leftrightarrow$$

$$2\big((s^3 - 12Rrs)^2 - 2((4Rr + r^2)^3 - 12Rr^3s^2) + 6r^4s^2\big) \overset{?}{\geq}$$
$$3\big(r^2s(s^3 - 12Rrs) + (4Rr + r^2)^3 - 12Rr^3s^2\big) \Leftrightarrow 2s^6 - (48Rr + 3r^2)s^4 +$$
$$r^2(288R^2 + 120Rr + 12r^2)s^2 - r^3(448R^3 + 336R^2r + 84Rr^2 + 7r^3) \boxed{\overset{?}{\underset{(*)}{\geq}}} 0 \text{ and } \because$$

$$2(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \overset{?}{\geq} 2(s^2-16Rr+5r^2)^3 \Leftrightarrow (48R-33r)s^4 -$$

$$r(1248R^2-1080Rr+138r^2)s^2 + r^2(7744R^3-8016R^2r+2316Rr^2-257r^3)$$

$$\boxed{\underset{(**)}{\overset{?}{\geq}}} 0 \text{ and } \because (48R-33r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**) \overset{?}{\geq} (48R-33r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (72R^2-144Rr+48r^2)s^2 \boxed{\underset{(***)}{\overset{?}{\geq}}} r(1136R^3-2028R^2r+1041Rr^2-142r^3)$$

$$\text{and finally, } \overbrace{(72R^2-144Rr+48r^2)}^{=72R(R-2r)+48r^2>0} s^2 \overset{\text{Rouche}}{\geq}$$

$$(72R^2-144Rr+48r^2)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right) \overset{?}{\geq}$$

$$r(1136R^3-2028R^2r+1041Rr^2-142r^3)$$

$$\Leftrightarrow (R-2r)(144R^3-356R^2r+200Rr^2-47r^3) \overset{?}{\geq}$$

$$2(R-2r).\sqrt{R^2-2Rr}.(72R^2-144Rr+48r^2)$$

$$\Leftrightarrow (144R^3-356R^2r+200Rr^2-47r^3)^2 \overset{?}{\geq} 4(R^2-2Rr)(72R^2-144Rr+48r^2)$$

$$\left(\begin{array}{c} \because R-2r \overset{\text{Euler}}{\geq} 0 \text{ and } \because 144R^3-356R^2r+200Rr^2-47r^3 = \\ (R-2r)(34R(R-2r)+110R^2+64r^2)+81r^3 \overset{\text{Euler}}{\geq} 81r^3 > 0 \end{array}\right)$$

$$\Leftrightarrow (4t+1)((2t-5)^2(148t(t-2)+140t^2+98)+156t-241) \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \forall \Delta XYZ_{s,R,r} \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \sum_{\text{cyc}} x^2 + 6 \geq \frac{3}{2}\left(\sum_{\text{cyc}} x + \sum_{\text{cyc}} \frac{1}{x}\right) \forall x,y,z>0 \,|\, xyz=1,$$

$$''='' \text{ iff } x=y=z=1 \text{ (QED)}$$

**2179.**

$$\text{If } x,y,z>0, \sum_{\text{cyc}} x = 1 \text{ and } \lambda \geq \frac{2}{3} \text{ then :}$$

$$\lambda \sum_{\text{cyc}} x^3 + xyz \geq \frac{3\lambda+1}{9} \sum_{\text{cyc}} x^2$$

*Proposed by Marin Chirciu-Romania*

***Solution by Soumava Chakraborty-Kolkata-India***

$$\lambda\sum_{cyc}x^3+xyz-\frac{3\lambda+1}{9}\sum_{cyc}x^2=$$

$$\frac{\lambda}{3}\left(3\sum_{cyc}x^3-\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)\right)+\frac{1}{9}\left(9xyz-\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)\right)\left(\because\sum_{cyc}x=1\right)$$

$$\overset{\lambda\geq\frac{2}{3}}{\geq}\frac{2}{9}\left(3\sum_{cyc}x^3-\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)\right)+\frac{1}{9}\left(9xyz-\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)\right)$$

$$\left(\because\sum_{cyc}x^3\overset{\text{Chebyshev}}{\geq}\frac{1}{3}\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)\right)$$

$$=\frac{1}{3}\left(2\sum_{cyc}x^3-\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)+3xyz\right)$$

$$=\frac{1}{3}\left(\sum_{cyc}x^3+3xyz-\sum_{cyc}x^2y-\sum_{cyc}xy^2\right)\overset{\text{Schur}}{\geq}0$$

$$\therefore\lambda\sum_{cyc}x^3+xyz\geq\frac{3\lambda+1}{9}\sum_{cyc}x^2\ \forall\,x,y,z>0\,\Big|\sum_{cyc}x=1\text{ and }\lambda\geq\frac{2}{3},$$

$$''='' \text{ iff } x=y=z=\frac{1}{3}\ (\text{QED})$$

**2180. If $a,b,c\in\mathbb{R}$ then:**

$$\sum_{cyc}\sqrt{a^2+ab+b^2}\geq\sqrt{4\sum_{cyc}a^2+5\sum_{cyc}ab}$$

***Proposed by Nguyen Hung Cuong-Vietnam***

***Solution by Jenish Rijal-Nepal***

$$\sqrt{a^2+ab+b^2}=\sqrt{\left(a+\frac{b}{2}\right)^2+\left(\frac{\sqrt{3}b}{2}\right)^2}$$

*Via Minkowski's Inequality, we have:*

$$\sum_{cyc}\sqrt{\left(a+\frac{b}{2}\right)^2+\left(\frac{\sqrt{3}b}{2}\right)^2} \overset{①}{\underset{Minkowski}{\geq}} \sqrt{\left(\sum_{cyc}\left(a+\frac{b}{2}\right)\right)^2+\left(\sum_{cyc}\frac{\sqrt{3}b}{2}\right)^2}$$

$$Also, \sum_{cyc}\left(a+\frac{b}{2}\right)=\frac{3}{2}(a+b+c) \quad \text{and} \quad \sum_{cyc}\frac{\sqrt{3}b}{2}=\frac{\sqrt{3}}{2}(a+b+c)$$

*Substituting these back into* ① :

$$LHS \geq \sqrt{\frac{9}{4}(a+b+c)^2+\frac{3}{4}(a+b+c)^2}=\sqrt{3(a+b+c)^2}=\sqrt{3\sum_{cyc}a^2+6\sum_{cyc}ab}$$

By the well – known inequality $\sum_{cyc}a^2 \geq \sum_{cyc}ab$ :

$$LHS \geq \sqrt{3\sum_{cyc}a^2+6\sum_{cyc}ab} \geq \sqrt{\left(4\sum_{cyc}a^2+5\sum_{cyc}ab\right)-\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)} \geq$$

$$\geq \sqrt{4\sum_{cyc}a^2+5\sum_{cyc}ab}$$

*Equality holds if and only if* $a=b=c$.

**2181.** If $a,b,c>0$ and $ab+bc+ca=3$ then prove that :

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca} \geq \frac{75}{16a^2b^2c^2+9}$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Let } ab=x, bc=y, ca=z \text{ and then : } \sum_{cyc}\frac{1}{bc} \overset{?}{\geq} \frac{75}{16a^2b^2c^2+9}$$

$$\Leftrightarrow \sum_{cyc}\frac{1}{x} \boxed{\overset{?}{\underset{(*)}{\geq}}} \frac{75}{16xyz+9} \text{ subject to } \sum_{cyc}x=3$$

Assigning $y+z=X, z+x=Y, x+y=Z \Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0 \Rightarrow X+Y>Z, Y+Z>X, Z+X>Y \Rightarrow X,Y,Z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say) and so, $2\sum_{cyc}x=\sum_{cyc}X=2s \Rightarrow \sum_{cyc}x \overset{(\bullet)}{=} s \Rightarrow x=s-X, y=s-Y, z=s-Z$

$$\therefore xyz \overset{(\bullet\bullet)}{=} r^2 s \text{ and hence } \sum_{cyc} xy = \sum_{cyc} (s-X)(s-Y) \Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2 \text{ and so,}$$

$$(*) \overset{x+y+z=3}{\Leftrightarrow} \frac{\sum_{cyc} xy}{xyz} \overset{?}{\geq} \frac{\frac{75}{9}\cdot\left(\sum_{cyc} x\right)^2}{\frac{9}{27}\cdot\left(\sum_{cyc} x\right)^3 + 16xyz} = \frac{25\left(\sum_{cyc} x\right)^2}{\left(\sum_{cyc} x\right)^3 + 48xyz}$$

$$\overset{\text{via } (\bullet),(\bullet\bullet) \text{ and } (\bullet\bullet\bullet)}{\Leftrightarrow} \frac{4Rr + r^2}{r^2 s} \overset{?}{\geq} \frac{25s^2}{s^3 + 48r^2 s} \Leftrightarrow (R-6r)s^2 + 12r^2(4R+r) \boxed{\underset{(**)}{\overset{?}{\geq}}} 0$$

and $(**)$ is trivially true if : $R - 6r \geq 0$ and so, we now focus on the case when :

$$R - 6r < 0 \text{ and then : LHS of } (**) \overset{\text{Rouche}}{\geq}$$

$$(R-6r)\left(2R^2 + 10Rr - r^2 + 2(R-2r).\sqrt{R^2 - 2Rr}\right) + 12r^2(4R+r) \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(2R^2 + 2Rr - 9r^2) \boxed{\underset{(***)}{\overset{?}{\geq}}} 2(6r-R)(R-2r).\sqrt{R^2 - 2Rr} \text{ and } \because$$

$$R - 2r \overset{\text{Euler}}{\geq} 0 \text{ and } \because 2R^2 + 2Rr - 9r^2 = 2(R^2 - 4r^2) + 2r(R-2r) + 3r^2 \overset{\text{Euler}}{\geq}$$

$3r^2 > 0 \therefore$ in order to prove $(***)$, it suffices to prove :

$$(2R^2 + 2Rr - 9r^2)^2 \overset{?}{>} 4(R^2 - 2Rr)(6r - R)^2$$

$$\Leftrightarrow 64t^3 - 272t^2 + 252t + 81 \overset{?}{>} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (4t+1)(4t-9)^2 \overset{?}{>} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \text{ is true } \forall\, \Delta\, XYZ_{s,R,r} \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq \frac{75}{16a^2b^2c^2 + 9} \forall\, a, b, c > 0 \,|\, ab + bc + ca = 3,$$

$$'' = '' \text{ iff } a = b = c = 1 \text{ (QED)}$$

**2182. If $x, y, z > 0$ then:**

$$\frac{x + 2\sqrt{yz}}{y + z} + \frac{y + 2\sqrt{zx}}{z + x} + \frac{z + 2\sqrt{xy}}{x + y} \geq 3$$

*Proposed by Gheorghe Crăciun-Romania*

*Solution 1 by Kasem Abotrabi, Khaled Abd Imouti-Syria*

$$L = \frac{x + 2\sqrt{yz}}{y + z} + \frac{y + 2\sqrt{zx}}{z + x} + \frac{z + 2\sqrt{xy}}{x + y} \geq 3$$

Suppose : $x = a^2, y = b^2, z = c^2 \Rightarrow \sqrt{xy} = ab, \sqrt{zx} = ac, \sqrt{yz} = bc$

$$SO : L = \frac{c^2 + 2ab}{a^2 + b^2} + \frac{b^2 + 2ac}{a^2 + c^2} + \frac{a^2 + 2cb}{c^2 + b^2} \geq 3$$

$$\frac{c^2+2ab}{a^2+b^2}=\frac{c^2+2ab-(a^2+b^2)+(a^2-b^2)}{a^2+b^2}=1+\frac{c^2-(a-b)^2}{a^2+b^2}$$

So let : $L=\frac{c^2-(a-b)^2}{a^2+b^2}+\frac{b^2-(c-a)^2}{c^2+a^2}+\frac{a^2-(b-c)^2}{b^2+c^2}\geq 0$

$$\frac{c^2}{a^2+b^2}=\frac{a^2+b^2+c^2-(a^2+b^2)}{a^2+b^2}=\frac{a^2+b^2+c^2}{a^2+b^2}-1$$

$$L=(a^2+b^2+c^2)\left(\frac{1}{a^2+b^2}+\frac{1}{c^2+a^2}+\frac{1}{b^2+c^2}\right)-3+\left(\frac{1}{a^2+b^2}+\frac{1}{c^2+a^2}+\frac{1}{b^2+c^2}\right)\geq 6$$

$$L=1+\frac{a^2+b^2+c^2+2ab}{a^2+b^2}+1+\frac{a^2+b^2+c^2+2ca}{a^2+c^2}+1+\frac{a^2+b^2+c^2+2cb}{b^2+c^2}\geq ?\,6+3$$

$$L=\frac{a^2+b^2+c^2+(a+b)^2}{a^2+b^2}+\frac{a^2+b^2+c^2+(a+c)^2}{a^2+c^2}+\frac{a^2+b^2+c^2+(b+c)^2}{b^2+c^2}\geq ?\,9$$

$$L\geq\frac{(a+b+c+a+b+a+b+c+a+c+a+b+c+b+c)^2}{2a^2+2b^2+2c^2}$$

$$L\geq\frac{25(a+b+c)^2}{2(a^2+b^2+c^2)}\geq\frac{25}{2}\cdot\frac{3}{2}=\frac{75}{4}\geq ?\,9\ \ True$$

**(Bergstrom's inequality)**

**Equality holds when a = b = c =1**

*Solution 2 by Kasem Abotrabi, Khaled Abd Imouti-Syria*

$$x,y,z>0$$

$$A=\frac{x+2\sqrt{yz}}{y+z}+\frac{y+2\sqrt{zx}}{z+x}+\frac{z+2\sqrt{xy}}{x+y}\geq ?\,3$$

$$A=\sum_{cyc}\frac{x}{y+z}+\sum_{cyc}\frac{2\sqrt{yz}}{y+z}$$

$$A=\sum_{cyc}\frac{(\sqrt{x})^2}{y+z}+\sum_{cyc}\frac{2\sqrt{yz}+y+z}{y+z}-3$$

$$A=\sum_{cyc}\frac{(\sqrt{x})^2}{y+z}+\sum_{cyc}\frac{(\sqrt{y}+\sqrt{z})^2}{y+z}-3\qquad(\text{Bergstrom's inequality})$$

$$A\geq\frac{(\sqrt{x}+\sqrt{y}+\sqrt{z})^2}{x+y+z}+\frac{(2(\sqrt{x}+\sqrt{y}+\sqrt{z}))^2}{x+y+z}-3$$

$$A\geq\frac{5\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)^2}{2(x+y+z)}-3\qquad .\ AM-GM$$

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\geq 3\sqrt[3]{\sqrt{xyz}} \quad ,x+y+z\geq 3\sqrt[3]{xyz}$$

$$A\geq \frac{5\left(3\sqrt[3]{\sqrt{xyz}}\right)^2}{2\left(3\sqrt{xyz}\right)}-3\Rightarrow A\geq \frac{45\sqrt[3]{xyz}}{6\sqrt[3]{xyz}}-3\Rightarrow$$

$$A\geq \frac{15}{2}-3\Rightarrow A\geq \frac{9}{2}$$

Equality holds when x = y = z =1

**2183.** **If $a^3b^3(1+a^4)(1+b^4)\geq 4$, then prove that :**

$$a^2+b^2\geq 2$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Since } ab\leq \frac{a^2+b^2}{2}\ \forall\, a,b\in\mathbb{R} \text{ and } xy\leq \frac{(x+y)^2}{4}\ \forall\, x,y\in\mathbb{R}$$

$$\therefore a^3b^3(1+a^4)(1+b^4)=ab(b^2+b^2a^4)(a^2+a^2b^4)\leq$$

$$\frac{a^2+b^2}{2}.(b^2+b^2a^4)(a^2+a^2b^4)\ (\because b^2+b^2a^4, a^2+a^2b^4>0)$$

$$\leq \frac{a^2+b^2}{2}.\frac{\left(a^2+b^2+a^2b^2(a^2+b^2)\right)^2}{4}$$

$$=\frac{a^2+b^2}{2}.\frac{(a^2+b^2)^2+a^4b^4(a^2+b^2)^2+2a^2b^2(a^2+b^2)^2}{4}$$

$$\leq \frac{a^2+b^2}{2}.\frac{(a^2+b^2)^2+\frac{(a^2+b^2)^4}{16}.(a^2+b^2)^2+\frac{(a^2+b^2)^2}{2}(a^2+b^2)^2}{4}$$

$$(\because (a^2+b^2)^2>0)\ \therefore 4\leq a^3b^3(1+a^4)(1+b^4)\leq \frac{t\left(t^2+\frac{t^6}{16}+\frac{t^4}{2}\right)}{8}\ (t=a^2+b^2)$$

$$\Rightarrow t^7+8t^5+16t^3-512\geq 0$$

$$\Rightarrow (t-2)\left(t^6+2t^5+12t^4+24t^3+64t^2+128t+256\right)\geq 0$$

$$\Rightarrow t=a^2+b^2\geq 2\ (\because t>0)\ \therefore a^2+b^2\geq 2 \text{ whenever } a^3b^3(1+a^4)(1+b^4)\geq 4$$

$$\text{and } a,b\in\mathbb{R}, ''='' \text{ iff } (a=b=1) \text{ or } (a=b=-1)\ (\text{QED})$$

**2184.** **If $a^2b^2(1+a^3)(1+b^3)\geq 4$ then prove that :**

$$a^2+b^2\geq 2$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Since } xy \le \frac{(x+y)^2}{4}\ \forall\, x,y \in \mathbb{R} \therefore a^2b^2(1+a^3)(1+b^3)$$
$$= (a^2+a^2b^3)(b^2+b^2a^3) \le \frac{\left(a^2+b^2+a^2b^2(a+b)\right)^2}{4}$$
$$= \frac{(a^2+b^2)^2 + a^4b^4(a+b)^2 + 2a^2b^2(a^2+b^2)(a+b)}{4} \le$$
$$\frac{(a^2+b^2)^2 + a^4b^4.2(a^2+b^2) + 2a^2b^2(a^2+b^2).\sqrt{2(a^2+b^2)}}{4}$$
$$(\because a^4b^4, a^2b^2(a^2+b^2) > 0) \le$$
$$\frac{(a^2+b^2)^2 + \frac{(a^2+b^2)^4}{16}.2(a^2+b^2) + \frac{(a^2+b^2)^2}{2}(a^2+b^2).\sqrt{2(a^2+b^2)}}{4}$$
$$(\because a^2+b^2 > 0)$$
$$\therefore 4 \le a^2b^2(1+a^3)(1+b^3) \le \frac{\frac{m^4}{4} + \frac{1}{16}.\frac{m^8}{16}.m^2 + \frac{1}{2}.\frac{m^4}{4}.\frac{m^2}{2}.m}{4}\ \left(m = \sqrt{2(a^2+b^2)}\right)$$
$$\Rightarrow m^{10} + 16m^7 + 64m^4 - 4096 \ge 0$$
$$\Rightarrow (m-2)\begin{pmatrix} m^9 + 2m^8 + 4m^7 + 24m^6 + 48m^5 + 96m^4 + 256m^3 + \\ 512m^2 + 1024m + 2048 \end{pmatrix} \ge 0$$
$$\Rightarrow m = \sqrt{2(a^2+b^2)} \ge 2\ (\because m \ge 0) \Rightarrow a^2+b^2 \ge 2$$
$$\text{whenever } a^2b^2(1+a^3)(1+b^3) \ge 4 \text{ and } a,b \in \mathbb{R}, '' ='' \text{ iff } a=b=1 \text{ (QED)}$$

**2185.** If $x,y,z > 0$ then prove that :

$$\sum_{cyc}\left((y+z).\sqrt{\frac{yz}{(x+y)(x+z)}}\right) \ge \sum_{cyc} x$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Assigning } y+z = a, z+x = b, x+y = c \Rightarrow a+b-c = 2z > 0,$$
$$b+c-a = 2x > 0 \text{ and } c+a-b = 2y > 0 \Rightarrow a+b > c, b+c > a, c+a > b$$
$$\Rightarrow a,b,c \text{ form sides of a triangle with semiperimeter, circumradius and inradius}$$
$$= s, R, r \text{ (say) yielding } 2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x = s$$

$$\Rightarrow x = s - a, y = s - b, z = s - c \therefore \sum_{cyc}\left((y+z).\sqrt{\frac{yz}{(x+y)(x+z)}}\right) \overset{?}{\geq} \sum_{cyc} x$$

$$\Leftrightarrow \sum_{cyc}\left(a.\sqrt{\frac{(s-b)(s-c)}{bc}}\right) \overset{?}{\geq} s \Leftrightarrow \sum_{cyc} a\sin\frac{A}{2} \underbrace{\overset{?}{\geq}}_{(*)} s$$

$$\text{Now, } \sum_{cyc} a\sin\frac{A}{2} = \sum_{cyc}\left(\frac{a\sin\frac{A}{2}\cos\frac{B-C}{2}}{\cos\frac{B-C}{2}}\right) \geq \sum_{cyc}\left(a\sin\frac{A}{2}\cos\frac{B-C}{2}\right)$$

$$\left(\because 0 < \cos\frac{B-C}{2} \leq 1 \text{ and analogs}\right) = \frac{1}{2}\sum_{cyc}\left(a(\cos B + \cos C)\right)$$

$$= \frac{1}{2}\sum_{cyc}\left(a\left(\sum_{cyc}\cos A - \cos A\right)\right) = \frac{R+r}{2R}(2s) - \frac{R}{2}\sum_{cyc}\sin 2A$$

$$= \frac{s(R+r)}{R} - \frac{R}{2}.4.\frac{4Rrs}{8R^3} = s \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\left((y+z).\sqrt{\frac{yz}{(x+y)(x+z)}}\right) \geq \sum_{cyc} x \ \forall\, x,y,z > 0, '' ='' \text{ iff } x = y = z \text{ (QED)}$$

**2186. If $a, b > 0; a + b = 2a^2b^2$ then:**

$$\frac{a+1}{b^2} + \frac{b+1}{a^2} \geq 4$$

***Proposed by Nguyen Hung Cuong – Vietnam***

***Solution by Daniel Sitaru – Romania***

**Denote $S = a + b; P = ab; S > 0; P > 0$**

$$a + b \overset{AM-GM}{\geq} 2\sqrt{ab} \Rightarrow S \geq 2\sqrt{P} \Rightarrow 2P^2 \geq 2\sqrt{P}$$

$$\Rightarrow P^2 \geq P \Rightarrow P^4 \geq P \Rightarrow P^3 \geq 1 \Rightarrow P \geq 1 \Rightarrow P - 1 \geq 0 \quad (1)$$

**We used the hypothesis: $S = 2P^2$.**

$$\frac{a+1}{b^2} + \frac{b+1}{a^2} \geq 4 \Leftrightarrow a^2(a+1) + b^2(b+1) \geq 4a^2b^2$$

$$a^3 + b^3 + a^2 + b^2 - 4a^2b^2 \geq 0$$

$$S^3-3SP+S^2-2P-4P^2\geq 0$$

$$8P^6-6P^3+4P^4-2P-4P^2\geq 0$$

$$4P^5-3P^2+2P^3-1-2P\geq 0$$

$$4P^5+2P^3-3P^2-2P-1\geq 0$$

$$4P^5-4P^4+4P^4-4P^3+6P^3-6P^2+3P^2-3P+P-1\geq 0$$

$$4P^4(P-1)+4P^3(P-1)+6P^2(P-1)+3P(P-1)+(P-1)\geq 0$$

$$(P-1)(4P^4+4P^3+6P^2+3P+1)\geq 0$$

$$P-1\geq 0 \quad \text{(True by (1))}$$

Equality holds for $a=b=1$.

**2187.** If $a,b>0;ab=a+b$ then:

$$\frac{a+1}{b^3}+\frac{b+1}{a^3}\geq\frac{3}{4}$$

*Proposed by Nguyen Hung Cuong – Vietnam*

*Solution by Daniel Sitaru – Romania*

Denote: $S=a+b;P=ab;S>0;P>0$

$$a+b\overset{AM-GM}{\geq}2\sqrt{ab}\Rightarrow S\geq 2\sqrt{P}\Rightarrow S\geq 2\sqrt{S}\Rightarrow$$

$$\Rightarrow S^2\geq 4S\Rightarrow S\geq 4\Rightarrow S-4\geq 0 \quad (1)$$

$$\frac{a+1}{b^3}+\frac{b+1}{a^3}\geq\frac{3}{4}\Leftrightarrow 4[a^3(a+1)+b^3(b+1)]\geq 3a^3b^3$$

$$4(a^4+b^4)+4(a^3+b^3)\geq 3a^3b^3$$

$$4[(S^2-2P)^2-2P^2]+4(S^3-3SP)-3P^3\geq 0$$

$$S=P\Rightarrow 4(S^2-2S)^2-8S^2+4S^3-12S^2-3S^3\geq 0$$

$$4S^4-16S^3+16S^2-8S^2+S^3-12S^2\geq 0$$

$$4S^4-15S^3-4S^2\geq 0\Leftrightarrow 4S^2-15S-4\geq 0$$

$$4S^2-16S+S-4\geq 0\Leftrightarrow 4S(S-4)+(S-4)\geq 0$$

$$(S-4)(4S+1)\geq 0\Leftrightarrow S-4\geq 0 \quad \text{(True by (1))}.$$

Equality holds for $a=b=2$.

**2188.** If $a,b,c\geq 0, ab+bc+ca=1$ then prove that :

$$\frac{a}{(a+1)^2}+\frac{b}{(b+1)^2}+\frac{c}{(c+1)^2}\geq\frac{1}{a+b+c}+\frac{abc}{4}$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

When exactly one variable equals to zero WLOG we may assume $a=0$ $(b,c>0\wedge bc=1)$, then it boils down to proving : $\frac{b}{(b+1)^2}+\frac{c}{(c+1)^2}\underset{(*)}{\overset{?}{\geq}}\frac{1}{b+c}$

$$\Leftrightarrow\frac{bc(b+c)+4bc+b+c}{(bc+b+c+1)^2}=\frac{2(b+c)+4}{(2+b+c)^2}=\frac{2}{b+c+2}\overset{?}{\geq}\frac{1}{b+c}$$

$\Leftrightarrow\boxed{b+c\overset{?}{\geq}2=2\sqrt{bc}}\ (\because bc=1)\rightarrow$ true via AM − GM and when : $a,b,c>0$,

then : $ab+bc+ca=1\Rightarrow 1-bc=a(b+c)\overset{AM-GM}{\geq}2a.\sqrt{bc}$

$$\Rightarrow\boxed{a\leq\frac{1-t^2}{2t}}\ (t=\sqrt{bc})\ \therefore\sqrt{bc}(a+1)=ta+t\leq\frac{1-t^2}{2}+t\overset{?}{<}2$$

$\Leftrightarrow t^2-2t+3\overset{?}{>}0\Leftrightarrow(t-1)^2+2\overset{?}{>}0\rightarrow$ true $\therefore\sqrt{bc}(a+1)<2\Rightarrow\frac{1}{(a+1)^2}>\frac{bc}{4}$

and so, $\boxed{\frac{a}{(a+1)^2}>\frac{abc}{4}}$ and $\frac{1}{a+b+c}\overset{a>0}{<}\frac{1}{b+c}\overset{\text{via }(*)}{\leq}\frac{b}{(b+1)^2}+\frac{c}{(c+1)^2}$

$$\Rightarrow\frac{a}{(a+1)^2}+\frac{b}{(b+1)^2}+\frac{c}{(c+1)^2}>\frac{1}{a+b+c}+\frac{abc}{4}\text{ and so,}$$

$$\frac{a}{(a+1)^2}+\frac{b}{(b+1)^2}+\frac{c}{(c+1)^2}\geq\frac{1}{a+b+c}+\frac{abc}{4}\ \forall a,b,c\geq 0\,|\,ab+bc+ca=1,$$

$$''='' \text{ iff } \begin{pmatrix}a=0\\b=c=1\end{pmatrix}\text{ or }\begin{pmatrix}b=0\\c=a=1\end{pmatrix}\text{ or }\begin{pmatrix}c=0\\a=b=1\end{pmatrix}\ \text{(QED)}$$

**2189.** If $a,b,c>0$ then:

$$\sum_{cyc}\frac{\sqrt{(1+b^2)(1+c^2)}}{a}\geq 6$$

*Proposed by Nguyen Hung Cuong – Vietnam*

*Solution by Daniel Sitaru – Romania*

$$\sum_{cyc}\frac{\sqrt{(1+b^2)(1+c^2)}}{a}\overset{AM-GM}{\geq}\sum_{cyc}\frac{\sqrt{2b\cdot 2c}}{a}=$$

$$=2\sum_{cyc}\frac{\sqrt{bc}}{a}\overset{AM-GM}{\geq}2\cdot 3\sqrt[3]{\frac{\sqrt{bc}}{a}\cdot\frac{\sqrt{ca}}{b}\cdot\frac{\sqrt{ab}}{c}}=6\sqrt[3]{\frac{abc}{abc}}=6\cdot 1=6$$

Equality holds for $a=b=c=1$.

**2190.** If $a,b>0, ab=1$ then:

$$a^{a^2+2b^2}\cdot b^{b^2+2a^2}\leq 1$$

*Proposed by Nguyen Hung Cuong – Vietnam*

*Solution by Daniel Sitaru – Romania*

$$a^{a^2+2b^2}\cdot b^{b^2+2a^2}\leq 1\Rightarrow \ln\left(a^{a^2+2b^2}\cdot b^{b^2+2a^2}\right)\leq \ln 1$$

$$(a^2+2b^2)\ln a+(b^2+2a^2)\ln b\leq 0$$

$$\left(a^2+\frac{2}{a^2}\right)\ln a+\left(\frac{1}{a^2}+2a^2\right)\ln\frac{1}{a}\leq 0$$

$$a^2\ln a+\frac{2}{a^2}\ln a+\frac{1}{a^2}\ln a^{-1}+2a^2\ln a^{-1}\leq 0$$

$$a^2\ln a+\frac{2}{a^2}\ln a-\frac{1}{a^2}\ln a-2a^2\ln a\leq 0$$

$$\left(a^2+\frac{2}{a^2}-\frac{1}{a^2}-2a^2\right)\ln a\leq 0$$

$$\left(\frac{1}{a^2}-a^2\right)\ln a\leq 0\Leftrightarrow\frac{a^4-1}{a^2}\ln a\geq 0$$

$$\frac{(a-1)(a+1)(a^2+1)}{a^2}\ln a\geq 0\Leftrightarrow (a-1)\ln a\geq 0$$

If $a\geq 1\Rightarrow a-1\geq 0;\ln a\geq 0\Rightarrow (a-1)\ln a\geq 0$

If $0<a\leq 1\Rightarrow a-1\leq 0;\ln a\leq 0\Rightarrow (a-1)\ln a\geq 0$

Equality holds for $a=b=1$.

**2191.** If $a,b,c>0$ and $ab+bc+ca=3$ then prove that :

$$4a^2b^2c^2+3\geq\frac{21abc}{a+b+c}$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Assigning } b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0,$$
$$y+z-x=2a>0 \text{ and } z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$$
$$\Rightarrow x,y,z \text{ form sides of a triangle XYZ with semiperimeter, circumradius and inradius}$$
$$= s,R,r\ (say); \text{ then}: \sum_{cyc} a = s, abc = r^2 s, \sum_{cyc} ab = 4Rr + r^2 \text{ and then}:$$
$$4a^2b^2c^2 + 3 \overset{?}{\geq} \frac{21abc}{a+b+c} \Leftrightarrow 4a^2b^2c^2 + \frac{3}{27}\left(\sum_{cyc} ab\right)^3 \overset{?}{\geq} \frac{21abc}{a+b+c}\cdot\frac{1}{9}\left(\sum_{cyc} ab\right)^2$$
$$\left(\because \sum_{cyc} ab = 3\right) \Leftrightarrow s((4Rr+r^2)^3 + 36s^2r^4) \overset{?}{\geq} 21r^2 s(4Rr+r^2)^2$$
$$\Leftrightarrow 16R^3 - 72R^2r - 39Rr^2 - 5r^3 + 9rs^2 \underset{(*)}{\overset{?}{\geq}} 0$$
$$\text{Now, since } 9rs^2 \overset{Gerretsen}{\geq} 9r(16Rr - 5r^2) \therefore \text{LHS of } (*) \geq$$
$$16R^3 - 72R^2r - 39Rr^2 - 5r^3 + 9r(16Rr - 5r^2) \overset{?}{\geq} 0$$
$$\Leftrightarrow 16R^3 - 72R^2r + 105Rr^2 - 50r^3 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(4R-5r)^2 \overset{?}{\geq} 0 \to \text{true}$$
$$\because R \overset{Euler}{\geq} 2r \Rightarrow (*) \text{ is true } \forall \Delta XYZ_{s,R,r} \therefore 4a^2b^2c^2 + 3 \geq \frac{21abc}{a+b+c}$$
$$\forall a,b,c>0 \,\middle|\, ab+bc+ca=3, \text{"="} \text{ iff } a=b=c=1 \text{ (QED)}$$

**2192.** **If $a,b,c>0; abc=1$ then:**

$$(a^3+1)^a(b^3+1)^b(c^3+1)^c \geq 8$$

*Proposed by Nguyen Hung Cuong – Vietnam*

*Solution by Daniel Sitaru – Romania*

$$(a^3+1)^a(b^3+1)^b(c^3+1)^c \geq 8$$
$$\ln\left[(a^3+1)^a(b^3+1)^b(c^3+1)^c\right] \geq \ln 8$$
$$a\ln(a^3+1) + b\ln(b^3+1) + c\ln(c^3+1) \geq 3\ln 2 \quad \text{(to prove)}$$
$$\text{Let be } f:(0,\infty)\to\mathbb{R}; f(x) = x\ln(1+x^3)$$
$$f'(x) = \ln(1+x^3) + x\cdot\frac{3x^2}{1+x^3} = \ln(1+x^3) + \frac{3x^3}{1+x^3} > 0$$
$$f'(x) > 0 \Rightarrow f \text{ increasing}$$

$$f''(x)=\frac{3x^2}{1+x^3}+\frac{9x^2(1+x^3)-3x^3\cdot 3x^2}{(1+x^3)^2}=\frac{3x^2}{1+x^3}+\frac{9x^2}{(1+x^3)^2}>0$$

$$f''(x)>0\Rightarrow f\text{ convex}$$

$$a\ln(a^3+1)+b\ln(b^3+1)+c\ln(c^3+1)=$$

$$=f(a)+f(b)+f(c)\overset{JENSEN}{\geq}3f\left(\frac{a+b+c}{3}\right)\overset{AM-GM}{\geq}$$

$$\geq 3f\left(\sqrt[3]{abc}\right)=3f\left(\sqrt[3]{1}\right)=3f(1)=3\ln 2$$

Equality holds for $a=b=c=1$.

**2193.** **If $a,b,c>0, n\in\mathbb{N}, \sum_{cyc}a^2\geq 3$ then:**

$$\sum_{cyc}\left(\frac{a^4}{a^2+2b+2c}\right)^n\geq\frac{3}{5^n}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Jenish Rijal-Nepal*

$$a^2+1\geq 2a\ \Rightarrow\sum_{cyc}2a\leq\sum_{cyc}(a^2+1)\ \Rightarrow 2\sum_{cyc}a\leq\sum_{cyc}a^2\ +3$$

$$\sum_{cyc}\frac{a^4}{a^2+2b+2c}\underbrace{\geq}_{\text{BERGSTROM}}\frac{(\sum a^2)^2}{\sum(a^2+2b+2c)}=\frac{(\sum a^2)^2}{\sum a^2+4\sum a}\geq$$

$$\geq\frac{(\sum a^2)^2}{\sum a^2+2\sum a^2+6}=\frac{(\sum a^2)^2}{3\sum a^2\ +6}\underbrace{\geq}_{6\leq 2\sum a^2}\frac{(\sum a^2)^2}{3\sum a^2\ +2\sum a^2}=\frac{\sum a^2}{5}\underbrace{\geq}_{\sum a^2\geq 3}=\frac{3}{5}$$

$$\frac{1}{3}\cdot\sum_{cyc}\left(\frac{a^4}{a^2+2b+2c}\right)^n\underbrace{\geq}_{\text{Power Mean}}\left(\frac{1}{3}\cdot\sum_{cyc}\frac{a^4}{a^2+2b+2c}\right)^n\geq\left(\frac{1}{3}\cdot\frac{3}{5}\right)^n=\frac{1}{5^n}$$

$$\sum_{cyc}\left(\frac{a^4}{a^2+2b+2c}\right)^n\geq\frac{3}{5^n}$$

*Equality holds if and only if $a=b=c=1$.*

**2194.** **If $a,b,c>0,\sum_{cyc}a^2=3abc$ and $n\in\mathbb{N}^*$ then :**

$$\sum_{cyc}\frac{a^n}{b^{n+1}c^{n+1}}\ge\frac{9}{a+b+c}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum_{cyc}\frac{a^n}{b^{n+1}c^{n+1}}=\sum_{cyc}\frac{\left(\frac{a}{bc}\right)^{n+1}}{a}\overset{Holder}{\ge}\frac{\left(\sum_{cyc}\frac{a}{bc}\right)^{n+1}}{3^{n-1}.\left(\sum_{cyc}a\right)}=\frac{3^{n+1}}{3^{n-1}.\left(\sum_{cyc}a\right)}$$

$$\left(\because\sum_{cyc}a^2=3abc\Rightarrow\sum_{cyc}\frac{a}{bc}=3\right)=\frac{9}{a+b+c}\text{ and so, }\sum_{cyc}\frac{a^n}{b^{n+1}c^{n+1}}\ge\frac{9}{a+b+c}$$

$$\forall\, a,b,c>0\,\Big|\sum_{cyc}a^2=3abc,''=''\text{ iff } a=b=c=1\text{ (QED)}$$

**2195.** If $x,y,z>0,\sum_{cyc}xy=3$ and $\frac{9}{4}\le\lambda\le 6$ then :

$$\sum_{cyc}x^3+\lambda xyz\ge\lambda+3$$

*Proposed by Marin Chirciu-Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\left(\sum_{cyc}x\right)^2\ge 3\sum_{cyc}xy=9\Rightarrow\sum_{cyc}x\ge 3\Rightarrow\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\ge 9$$

$$\Rightarrow\sum_{cyc}x^3+\lambda xyz-(\lambda+3)\ge\sum_{cyc}x^3+\lambda xyz-\left(\frac{\lambda+3}{9}\right)\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)$$

$$=\sum_{cyc}x^3-\frac{1}{3}\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-\frac{\lambda}{9}\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-9xyz\right)$$

$$\overset{\lambda\le 6}{\ge}\sum_{cyc}x^3-\frac{1}{3}\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-\frac{6}{9}\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-9xyz\right)$$

$$\left(\because\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\overset{AM-GM}{\geq}9xyz\right)$$

$$=\sum_{cyc}x^3-\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)+6xyz=\sum_{cyc}x^3+3xyz-\sum_{cyc}x^2y-\sum_{cyc}xy^2\overset{Schur}{\geq}0$$

$$\therefore\sum_{cyc}x^3+\lambda xyz\geq\lambda+3\ \forall\ x,y,z>0\,\Big|\sum_{cyc}xy=3\text{ and }\frac{9}{4}\leq\lambda\leq 6,$$

$$''='' \text{ iff } x=y=z=1\ (QED)$$

**2196.** **If $a,b,c>0; abc=1$ then:**

$$\frac{(a+2)^2}{a^3}+\frac{(b+2)^2}{b^3}+\frac{(c+2)^2}{c^3}\geq 27$$

*Proposed by Nguyen Hung Cuong – Vietnam*

*Solution by Daniel Sitaru – Romania*

$$\frac{(a+2)^2}{a^3}+\frac{(b+2)^2}{b^3}+\frac{(c+2)^2}{c^3}\overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{\left((a+2)(b+2)(c+2)\right)^2}{(abc)^2}}=3\sqrt[3]{\left((a+2)(b+2)(c+2)\right)^2}$$

**Remains to prove:**

$$3\sqrt[3]{\left((a+2)(b+2)(c+2)\right)^2}\geq 27\Leftrightarrow\sqrt[3]{\left((a+2)(b+2)(c+2)\right)^2}\geq 9$$

$$\left((a+2)(b+2)(c+2)\right)^2\geq 9^3(a+2)(b+2)(c+2)\geq 3^3$$

$$abc+2(ab+bc+ca)+4(a+b+c)+8\geq 27$$

$$2(ab+bc+ca)+4(a+b+c)\geq 18$$

$$ab+bc+ca+2(a+b+c)\geq 9\qquad\text{(to prove)}$$

$$ab+bc+ca+2(a+b+c)\overset{AM-GM}{\geq}9\sqrt[9]{ab\cdot bc\cdot ca\cdot a^2\cdot b^2\cdot c^2}=9\sqrt[3]{(abc)^4}=9\cdot 1=9$$

**Equality holds for $a=b=c=1$.**

**2197.** *Let $a_1,\ a_2,\ldots\ldots,a_n(n\geq 3)$ be positive real numbers such that*

$$\sum_{i=1}^{n}a_i^3\leq n.\ \textit{Prove that:}$$

$$\sum_{i=1}^{n}\sqrt{7a_i^2-12a_i+9}\geq 2\sum_{i=1}^{n}a_i^2$$

*Proposed by Nguyen Van Hoa-Vietnam*

*Solution by Jenish Rijal-Nepal*

$$Claim:\ \sqrt{7a^2-12a+9}\geq 2a^2-\frac{7}{6}(a^3-1)$$

$$Proof:\ \sqrt{7a^2-12a+9}-2a^2+\frac{7}{6}(a^3-1)=$$

$$=\left(\sqrt{7a^2-12a+9}-\frac{a+3}{2}\right)+\left(\frac{a+3}{2}-2a^2+\frac{7}{6}(a^3-1)\right)$$

$$\sqrt{7a^2-12a+9}-\frac{a+3}{2}=\frac{(7a^2-12a+9)-\frac{(a+3)^2}{4}}{\sqrt{7a^2-12a+9}+\frac{a+3}{2}}=$$

$$=\frac{27(a-1)^2}{4\left(\sqrt{7a^2-12a+9}+\frac{a+3}{2}\right)}$$

$$\frac{a+3}{2}-2a^2+\frac{7}{6}(a^3-1)=\frac{7}{6}a^3-2a^2+\frac{1}{2}a+\frac{1}{3}=\frac{(a-1)^2(7a+2)}{6}$$

$$\sqrt{7a^2-12a+9}-2a^2+\frac{7}{6}(a^3-1)=(a-1)^2\left[\frac{27}{4\left(\sqrt{7a^2-12a+9}+\frac{a+3}{2}\right)}+\frac{(7a+2)}{6}\right]\geq 0$$

$$\therefore \sqrt{7a^2-12a+9}\geq 2a^2-\frac{7}{6}(a^3-1)$$

$$\Rightarrow \sum_{i=1}^{n}\sqrt{7a_i^2-12a_i+9}\geq 2\sum_{i=1}^{n}a_i^2-\frac{7}{6}\left(\sum_{i=1}^{n}a_i^3-n\right)$$

$$Since\ \sum_{i=1}^{n}a_i^3-n\leq 0, we\ have\ -\frac{7}{6}\left(\sum_{i=1}^{n}a_i^3-n\right)\geq 0:$$

$$\sum_{i=1}^{n}\sqrt{7a_i^2-12a_i+9}\geq 2\sum_{i=1}^{n}a_i^2-\frac{7}{6}\left(\sum_{i=1}^{n}a_i^3-n\right)\geq 2\sum_{i=1}^{n}a_i^2-0=2\sum_{i=1}^{n}a_i^2$$

$$Therefore:\ \sum_{i=1}^{n}\sqrt{7a_i^2-12a_i+9}\geq 2\sum_{i=1}^{n}a_i^2$$

$$Equality\ holds\ if\ and\ only\ if\ a_1=a_2=\ldots..=a_n=1.$$

**2198.** *If* $a,b,c\geq 1$, *prove that*:

$$\left(\log_{1+a^3}\left(b^2+b^{b^2}\right)\right)^3+\left(\log_{1+b^3}\left(c^2+c^{c^2}\right)\right)^3+\left(\log_{1+c^3}\left(a^2+a^{a^2}\right)\right)^3\geq$$

$$\geq\left(\log_{1+a^3}\left(b^2+b^{b^2}\right)\right)^2+\left(\log_{1+b^3}\left(c^2+c^{c^2}\right)\right)^2+\left(\log_{1+c^3}\left(a^2+a^{a^2}\right)\right)^2$$

*Proposed by Pavlos Trifon-Greece*

*Solution by Jenish Rijal-Nepal*

Lemma: $t^{t^2}+t^2\ge t^3+1\quad\forall\, t\ge 1$

$$t^{t^2}=\left(1+(t-1)\right)^{t^2}\overset{Bernoulli}{\overbrace{\ge}}1+t^2(t-1)\Leftrightarrow t^{t^2}+t^2\ge t^3+1\quad\forall\, t\ge 1$$

Let $x=\log_{1+a^3}\left(b^2+b^{b^2}\right)\overset{Lemma}{\overbrace{\ge}}\log_{1+a^3}(b^3+1)>0,$

$$y=\log_{1+b^3}\left(c^2+c^{c^2}\right)\ge\log_{1+b^3}(c^3+1)>0\ \ and$$

$$z=\log_{1+c^3}\left(a^2+a^{a^2}\right)\ge\log_{1+c^3}(a^3+1)>0$$

We have:

$$xyz\ge\log_{1+a^3}(b^3+1)\cdot\log_{1+b^3}(c^3+1)\cdot\log_{1+c^3}(a^3+1)=$$

$$=\frac{\ln(b^3+1)}{\ln(1+a^3)}\cdot\frac{\ln(c^3+1)}{\ln(1+b^3)}\cdot\frac{\ln(a^3+1)}{\ln(1+c^3)}=1$$

Thus, it suffices to prove that: $x^3+y^3+z^3\ge x^2+y^2+z^2$ with $xyz\ge 1$

WLOG assume that: $x\ge y\ge z\Rightarrow x^2\ge y^2\ge z^2$

$$x^3+y^3+z^3\overset{Chebyshev}{\overbrace{\ge}}\frac{1}{3}(x+y+z)(x^2+y^2+z^2)\overset{AM-GM}{\overbrace{\ge}}$$

$$\ge\sqrt[3]{xyz}(x^2+y^2+z^2)\overset{xyz\ge 1}{\overbrace{\ge}}x^2+y^2+z^2$$

Thus $x^3+y^3+z^3\ge x^2+y^2+z^2$ .

Equality holds if and only if $a=b=c=1$.

**2199.** If $a,b,c>0,ab+bc+ca=1$ then:

$$\frac{(ab+1)^2}{a^2+4ab+b^2}+\frac{(bc+1)^2}{b^2+4bc+c^2}+\frac{(ca+1)^2}{c^2+4ca+a^2}\ge\frac{8}{3}$$

*Proposed by Nguyen Hung Cuong – Vietnam*

*Solution by Daniel Sitaru – Romania*

$$a^2+b^2+c^2\ge ab+bc+ca=1$$

$$a^2+b^2+c^2\ge 1\qquad (1)$$

$$\frac{(ab+1)^2}{a^2+4ab+b^2}+\frac{(bc+1)^2}{b^2+4bc+c^2}+\frac{(ca+1)^2}{c^2+4ca+a^2}=\sum_{cyc}\frac{(ab+1)^2}{a^2+4ab+b^2}\overset{BERGSTROM}{\ge}$$

$$\ge\frac{(ab+1+bc+1+ca+1)^2}{2(a^2+b^2+c^2)+4(ab+bc+ca)}=\frac{(1+3)^2}{2(a^2+b^2+c^2)+4}=$$

$$=\frac{16}{2(a^2+b^2+c^2)+4}\ge\frac{8}{a^2+b^2+c^2+2}\overset{(1)}{\ge}\frac{8}{1+2}=\frac{8}{3}$$

Equality holds for $a=b=c=\frac{\sqrt{3}}{8}$.

**2200.** If $a,b>0;a+b=2$ then:

$$a^4+b^4+6ab\ge 8$$

*Proposed by Nguyen Hung Cuong – Vietnam*

*Solution by Daniel Sitaru – Romania*

Denote: $S=a+b;P=ab$

$$S=a+b=2;2=a+b\overset{AM-GM}{\ge}2\sqrt{ab}=2\sqrt{P}$$

$$2\ge 2\sqrt{P}\Rightarrow\sqrt{P}\le 1\Rightarrow P\le 1\Rightarrow P-1\le 0\Rightarrow P-4\le -3<0$$

$$P-1\le 0;P-4<0\Rightarrow(P-1)(P-4)\ge 0$$

$$P^2-5P+4\ge 0\Rightarrow 2P^2-10P+8\ge 0$$

$$16+4P^2-16P-2P^2+6P\ge 8$$

$$(4-2P)^2-2P^2+6P\ge 8$$

$$(S^2-2P)^2-2P^2+6P\ge 8$$

$$a^4+b^4+6ab\ge 8$$

Equality holds for $a=b=1$.

*It's nice to be important but more important it's to be nice.*

*At this paper works a TEAM.*

*This is RMM TEAM.*

*To be continued!*

*Daniel Sitaru*