ROMANIAN MATHEMATICAL MAGAZINE

NEW IDENTITIES AND INEQUALITIES IN TRIANGLE-(IV)

By Bogdan Fuștei-Romania

$AA_1=n_a$ (Nagel cevian from A).

$BA_1=s-c$ and $CA_1=s-b$ [1] (pag.43 Teorem 136) and $IA_1=x$; $a+b+c=2s$

From BIC triangle and Stewart theorem: $IC^2(s-c)+IB^2(s-b)=a(x^2+(s-c)(s-b))$

$IA^2=(s-a)^2+r^2$ (and analogous)[2]

$$[(s-c)^2+r^2](s-c)+[(s-b)^2+r^2](s-b)=a(x^2+(s-c)(s-b))$$

$$(s-c)^3+r^2(s-c)+(s-b)^3+r^2(s-b)=a(x^2+(s-c)(s-b))$$

$s-c+s-b=a$; $x^3+y^3=(x+y)^3-3xy(x+y)$; x,y:real numbers ;

$$(s-c)^3+(s-b)^3+ar^2=a(x^2+(s-c)(s-b))$$

$$(s-c)^3+(s-b)^3=[(s-c)+(s-b)]^3-3(s-c)(s-b)(s-c+s-b)=a^3-3a(s-c)(s-b)$$

$$a^3-3a(s-c)(s-b)+ar^2=a(x^2+(s-c)(s-b))$$

$$a^2-3(s-c)(s-b)+r^2=x^2+(s-c)(s-b)$$

$$x^2=a^2-4(s-c)(s-b)+r^2$$

$$(s-c)(s-b)=\frac{(a+c-b)(a+b-c)}{4}\rightarrow 4(s-c)(s-b)=(a-(b-c))(a+(b-c))=a^2-(b-c)^2$$

$$x^2=a^2-a^2+(b-c)^2+r^2=(b-c)^2+r^2$$

$$IA_1=x=\sqrt{(b-c)^2+r^2}\text{ (and analogous);}$$

We consider AIA_1 triangle with $AA_1=n_a$;From triangle inequality :

$$\mathbf{n_a\le AI+\sqrt{(b-c)^2+r^2}}\textbf{ (and analogous) (1)}$$

ΔABC with $S=sr=\frac{1}{2}bc\sin A$; $\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}\rightarrow r=\frac{bc}{s}\sin\frac{A}{2}\cos\frac{A}{2}$;

$$AI=\frac{r}{\sin\frac{A}{2}}\rightarrow AI=\frac{bc}{s}\cos\frac{A}{2}\text{ (and analogous);}$$

Is well-known that: $l_a=\frac{2bc}{b+c}\cos\frac{A}{2}=\frac{2s}{b+c}\frac{bc}{s}\cos\frac{A}{2}=\frac{2s}{b+c}AI\rightarrow AI=\frac{b+c}{2s}l_a$ (and analogous);

From (1) and $AI=\frac{b+c}{2s}l_a$ we obtain:

$$\mathbf{\frac{b+c}{2s}\ge\frac{n_a-\sqrt{(b-c)^2+r^2}}{l_a}}\textbf{ (and analogous)(2)}$$

From (2) after summation:

$$\mathbf{\frac{2(a+b+c)}{2s}\ge\sum\frac{n_a-\sqrt{(b-c)^2+r^2}}{l_a}\rightarrow 2\ge\sum\frac{n_a-\sqrt{(b-c)^2+r^2}}{l_a}}\ \ (\mathbf{3})$$

From $2S=2sr=ah_a\rightarrow(a+b+c)r=ah_a\rightarrow 1+\frac{b+c}{a}=\frac{h_a}{r}\rightarrow\frac{b+c}{a}=\frac{h_a-r}{r}$ (and analogous);

$$\frac{AI}{r}=\frac{b+c}{a}\frac{l_a}{h_a}\text{ (and analogous)}\rightarrow\frac{b+c}{a}\frac{l_a}{h_a}\ge\frac{n_a-\sqrt{(b-c)^2+r^2}}{r}\rightarrow\frac{h_a-r}{r}\frac{l_a}{h_a}\ge\frac{n_a-\sqrt{(b-c)^2+r^2}}{r}$$

$$\frac{\mathbf{l_a}}{\mathbf{h_a}} \geq \frac{\mathbf{n_a}-\sqrt{(\mathbf{b}-\mathbf{c})^2+\mathbf{r}^2}}{\mathbf{h_a}-\mathbf{r}}\ \textbf{(and analogous) (4)}$$

Also $\frac{n_a}{h_a} = \frac{\sqrt{(b-c)^2+4r^2}}{2r}$ (and analogous)[3] $\rightarrow \frac{h_a}{r} = \frac{2n_a}{\sqrt{(b-c)^2+4r^2}} \rightarrow \frac{h_a-r}{r} = \frac{2n_a-\sqrt{(b-c)^2+4r^2}}{\sqrt{(b-c)^2+4r^2}}$

From : $\frac{h_a-r}{r}\frac{l_a}{h_a} \geq \frac{n_a-\sqrt{(b-c)^2+r^2}}{r}$ and $\frac{h_a-r}{r} = \frac{2n_a-\sqrt{(b-c)^2+4r^2}}{\sqrt{(b-c)^2+4r^2}} \rightarrow \frac{2n_a-\sqrt{(b-c)^2+4r^2}}{\sqrt{(b-c)^2+4r^2}}\frac{l_a}{h_a} \geq \frac{n_a-\sqrt{(b-c)^2+r^2}}{r}$

$$\frac{\mathbf{l_a}}{\mathbf{h_a}}\mathbf{r} \geq \frac{\sqrt{(\mathbf{b}-\mathbf{c})^2+4\mathbf{r}^2}\left(\mathbf{n_a}-\sqrt{(\mathbf{b}-\mathbf{c})^2+\mathbf{r}^2}\right)}{2\mathbf{n_a}-\sqrt{(\mathbf{b}-\mathbf{c})^2+4\mathbf{r}^2}}\ \textbf{(and analogous) (5)}$$

$l_a = AI + y$ and $AI = \frac{b+c}{2s}l_a \rightarrow y = \frac{a}{2s}l_a$; $2sr = ah_a \rightarrow \frac{a}{2s} = \frac{r}{h_a} \rightarrow y = \frac{l_a}{h_a}r$

$l_a = AI + \frac{l_a}{h_a}r$ (and analogous) and form (1) and (5) :

$$l_a \geq n_a - \sqrt{(b-c)^2 + r^2} + \frac{\sqrt{(b-c)^2+4r^2}\left(n_a-\sqrt{(b-c)^2+r^2}\right)}{2n_a-\sqrt{(b-c)^2+4r^2}}$$

$$\mathbf{l_a} \geq \left(\mathbf{n_a} - \sqrt{(\mathbf{b}-\mathbf{c})^2 + \mathbf{r}^2}\right)\left(\mathbf{1} + \frac{\sqrt{(\mathbf{b}-\mathbf{c})^2+4\mathbf{r}^2}}{2\mathbf{n_a}-\sqrt{(\mathbf{b}-\mathbf{c})^2+4\mathbf{r}^2}}\right)\ \textbf{(and analogous)(6)}$$

From (6) after summation:

$$\sum \frac{\mathbf{l_a}}{\mathbf{n_a}-\sqrt{(\mathbf{b}-\mathbf{c})^2+\mathbf{r}^2}} \geq \mathbf{3}+\sum \frac{\sqrt{(\mathbf{b}-\mathbf{c})^2+4\mathbf{r}^2}}{2\mathbf{n_a}-\sqrt{(\mathbf{b}-\mathbf{c})^2+4\mathbf{r}^2}}\ \textbf{(7)}$$

From (7) , $\frac{h_a-r}{r} = \frac{2n_a-\sqrt{(b-c)^2+4r^2}}{\sqrt{(b-c)^2+4r^2}}$ and $\frac{b+c}{a} = \frac{h_a-r}{r}$ we obtain:

$$\sum \frac{\mathbf{l_a}}{\mathbf{n_a}-\sqrt{(\mathbf{b}-\mathbf{c})^2+\mathbf{r}^2}} \geq \mathbf{3} + \frac{\mathbf{a}}{\mathbf{b}+\mathbf{c}} + \frac{\mathbf{b}}{\mathbf{a}+\mathbf{c}} + \frac{\mathbf{c}}{\mathbf{a}+\mathbf{b}}\ \textbf{(8)}$$

From (6) :

$$\frac{\mathbf{l_a}+\sqrt{(\mathbf{b}-\mathbf{c})^2+\mathbf{r}^2}-\mathbf{n_a}}{\mathbf{n_a}-\sqrt{(\mathbf{b}-\mathbf{c})^2+\mathbf{r}^2}} \geq \frac{\mathbf{a}}{\mathbf{b}+\mathbf{c}}\ \textbf{(and analogous)(9)}$$

From $\frac{AI}{r} = \frac{b+c}{a}\frac{l_a}{h_a} = \frac{1}{\sin\frac{A}{2}} \rightarrow \frac{1}{\sin\frac{A}{2}} \geq \frac{b+c}{a} \rightarrow \frac{a}{b+c} \geq \sin\frac{A}{2}$

From (9) and $\frac{a}{b+c} \geq \sin\frac{A}{2}$ we obtain :

$$\frac{\mathbf{l_a}+\sqrt{(\mathbf{b}-\mathbf{c})^2+\mathbf{r}^2}-\mathbf{n_a}}{\mathbf{n_a}-\sqrt{(\mathbf{b}-\mathbf{c})^2+\mathbf{r}^2}} \geq \sin\frac{\mathbf{A}}{\mathbf{2}}\ \textbf{(and analogous) (10)}$$

From (10) after summation:

$$\sum \frac{\mathbf{l_a}+\sqrt{(\mathbf{b}-\mathbf{c})^2+\mathbf{r}^2}-\mathbf{n_a}}{\mathbf{n_a}-\sqrt{(\mathbf{b}-\mathbf{c})^2+\mathbf{r}^2}} \geq \sin\frac{\mathbf{A}}{\mathbf{2}} + \sin\frac{\mathbf{B}}{\mathbf{2}} + \sin\frac{\mathbf{C}}{\mathbf{2}}\ \textbf{(11)}$$

Is easy to proof that: $\frac{R}{2r}=\frac{r_ar_br_c}{h_ah_bh_c}$; $\sin\frac{A}{2}=\sqrt{\frac{rr_a}{2Rh_a}}=\sqrt{\frac{r_a-r}{4R}}$ (and analogous) $\to$

$\frac{r_a}{h_a}=\frac{r_a-r}{2r}$ (and analogous);From (1) :

$$\frac{AI}{r}\geq\frac{n_a-\sqrt{(b-c)^2+r^2}}{r}\to\sqrt{\frac{2Rh_a}{rr_a}}\geq\frac{n_a-\sqrt{(b-c)^2+r^2}}{r}\to\frac{\sqrt{2Rr}}{n_a-\sqrt{(b-c)^2+r^2}}\geq\sqrt{\frac{r_a}{h_a}}\text{ (and analogous)(12)}$$

From

$$\frac{r_a}{h_a}=\frac{r_a-r}{2r}\text{ and (12)}\to$$

$$\frac{\sqrt{R}}{n_a-\sqrt{(b-c)^2+r^2}}\geq\frac{\sqrt{r_a-r}}{2r}\text{ (and analogous)(13)}$$

From $\frac{AI}{r}=\frac{1}{\sin\frac{A}{2}}=\sqrt{\frac{2R}{r}}\sqrt{\frac{h_a}{r_a}}\to\frac{AI}{r}=\sqrt{4\frac{r_ar_br_c}{h_ah_bh_c}}\sqrt{\frac{h_a}{r_a}}=2\sqrt{\frac{r_br_c}{h_bh_c}}$ (and analogous)

From $r_a+r_b+r_c=4R+r$ and $\frac{r_a}{h_a}=\frac{r_a-r}{2r}$ (and analogous) $\to\sum\frac{r_a}{h_a}=\frac{2R-r}{r}$

$$\left(\sum\sqrt{\frac{r_a}{h_a}}\right)^2=\sum\frac{r_a}{h_a}+2\sum\sqrt{\frac{r_br_c}{h_bh_c}}=\frac{2R-r+AI+BI+CI}{r}\to\sum\sqrt{\frac{r_a}{h_a}}=\sqrt{\frac{2R-r+AI+BI+CI}{r}}\text{ (14)}$$

From (12) after summation:

$$\sum\frac{\sqrt{2Rr}}{n_a-\sqrt{(b-c)^2+r^2}}\geq\sum\sqrt{\frac{r_a}{h_a}}\text{ (15)}$$

From (14) and (15) :

$$\sum\frac{\sqrt{2Rr}}{n_a-\sqrt{(b-c)^2+r^2}}\geq\sqrt{\frac{2R-r+AI+BI+CI}{r}}\text{ (16)}$$

From (16) :

$$\sum\frac{r}{n_a-\sqrt{(b-c)^2+r^2}}\geq\sqrt{\frac{2R-r+AI+BI+CI}{2R}}\text{ (17)}$$

From $\frac{r_a}{h_a}=\frac{r_a-r}{2r}$ (and analogous) and (14) :

$$\sum\sqrt{\frac{r_a-r}{r}}=\sqrt{\frac{2(2R-r+AI+BI+CI)}{r}}\text{ (18)}$$

$$\sum\sqrt{r_a-r}=\sqrt{2(2R-r+AI+BI+CI)}\text{ (19)}$$

From $\frac{r_a}{r}=\frac{n_a}{n_a-\sqrt{(b-c)^2+4r^2}}$ (and analogous)[4] and (18):

$$\sum\sqrt{\frac{n_a}{n_a-\sqrt{(b-c)^2+4r^2}}-1}=\sqrt{\frac{2(2R-r+AI+BI+CI)}{r}}\text{ (20)}$$

From (16) and (20):

$$2\sum\frac{\sqrt{Rr}}{n_a-\sqrt{(b-c)^2+r^2}}\geq\sum\sqrt{\frac{n_a}{n_a-\sqrt{(b-c)^2+4r^2}}-1}\text{ (21)}$$

From $s^2=n_a^2+2r_ah_a$(and analogous)[5] $\to n_a^2=s^2-2r_ah_a=\left(s-\sqrt{2r_ah_a}\right)\left(s+\sqrt{2r_ah_a}\right)\to \frac{n_a^2}{s-\sqrt{2r_ah_a}}=s++\sqrt{2r_ah_a}$

$2sr=ah_a=2S\to \frac{s}{h_a}=\frac{a}{2r}$ (and analogous);

$$\frac{n_a^2}{h_a\left(s-\sqrt{2r_ah_a}\right)}=\frac{a}{2r}+\sqrt{\frac{2r_a}{h_a}}\ \textbf{(and analogous)(22)}$$

From(22) after summation:

$$\sum\frac{n_a^2}{h_a\left(s-\sqrt{2r_ah_a}\right)}=\frac{s}{r}+\sqrt{\frac{2r_a}{h_a}}+\sqrt{\frac{2r_a}{h_a}}+\sqrt{\frac{2r_a}{h_a}}\ \textbf{(23)}$$

From (14) and (23):

$$\sum\frac{n_a^2}{h_a\left(s-\sqrt{2r_ah_a}\right)}=\frac{s}{r}+\sqrt{\frac{2(2R-r+AI+BI+CI)}{r}}\ \textbf{(24)}$$

From (20) and (24):

$$\sum\frac{n_a^2}{h_a\left(s-\sqrt{2r_ah_a}\right)}=\frac{s}{r}+\sum\sqrt{\frac{n_a}{n_a-\sqrt{(b-c)^2+4r^2}}-1}\ \textbf{(25)}$$

From (23) and $\frac{r_a}{h_a}=\frac{r_a-r}{2r}$ (and analogous):

$$\sum\frac{n_a^2}{h_a\left(s-\sqrt{2r_ah_a}\right)}=\frac{s}{r}+\sqrt{\frac{r_a-r}{r}}+\sqrt{\frac{r_a-r}{r}}+\sqrt{\frac{r_a-r}{r}}\ \textbf{(26)}$$

If we use Bergstrom Inequality ,we obtain:

$$\sum\frac{n_a^2}{h_a\left(s-\sqrt{2r_ah_a}\right)}\ge\frac{(n_a+n_b+n_c)^2}{s(h_a+h_b+h_c)-\left(h_a\sqrt{2r_ah_a}+h_b\sqrt{2r_bh_b}+h_c\sqrt{2r_ch_c}\right)}\ \textbf{(B)}$$

From (B)and (23),(24),(25),(26) we obtain:

$$\frac{s}{r}+\sqrt{\frac{2r_a}{h_a}}+\sqrt{\frac{2r_a}{h_a}}+\sqrt{\frac{2r_a}{h_a}}\ge\frac{(n_a+n_b+n_c)^2}{s(h_a+h_b+h_c)-\left(h_a\sqrt{2r_ah_a}+h_b\sqrt{2r_bh_b}+h_c\sqrt{2r_ch_c}\right)}\ \textbf{(27)}$$

$$\frac{s}{r}+\sqrt{\frac{2(2R-r+AI+BI+CI)}{r}}\ge\frac{(n_a+n_b+n_c)^2}{s(h_a+h_b+h_c)-\left(h_a\sqrt{2r_ah_a}+h_b\sqrt{2r_bh_b}+h_c\sqrt{2r_ch_c}\right)}\ \textbf{(28)}$$

$$\frac{s}{r}+\sum\sqrt{\frac{n_a}{n_a-\sqrt{(b-c)^2+4r^2}}-1}\ge\frac{(n_a+n_b+n_c)^2}{s(h_a+h_b+h_c)-\left(h_a\sqrt{2r_ah_a}+h_b\sqrt{2r_bh_b}+h_c\sqrt{2r_ch_c}\right)}\ \textbf{(29)}$$

$$\frac{s}{r}+\sqrt{\frac{r_a-r}{r}}+\sqrt{\frac{r_a-r}{r}}+\sqrt{\frac{r_a-r}{r}}\ge\frac{(n_a+n_b+n_c)^2}{s(h_a+h_b+h_c)-\left(h_a\sqrt{2r_ah_a}+h_b\sqrt{2r_bh_b}+h_c\sqrt{2r_ch_c}\right)}\ \textbf{(30)}$$

$\sum \frac{\sqrt{2Rr}}{n_a-\sqrt{(b-c)^2+r^2}} \geq \sqrt{\frac{2R-r+AI+BI+CI}{r}} \rightarrow \sum \frac{2\sqrt{Rr}}{n_a-\sqrt{(b-c)^2+r^2}} \geq \sqrt{\frac{2(2R-r+AI+BI+CI)}{r}}$ and (24):

$$\frac{s}{r}+2\sum \frac{\sqrt{Rr}}{n_a-\sqrt{(b-c)^2+r^2}} \geq \sum \frac{n_a^2}{h_a\left(s-\sqrt{2r_ah_a}\right)} \ \textbf{(31)}$$

From (31) and (B):

$$\frac{s}{r}+2\sum \frac{\sqrt{Rr}}{n_a-\sqrt{(b-c)^2+r^2}} \geq \frac{(n_a+n_b+n_c)^2}{s(h_a+h_b+h_c)-\left(h_a\sqrt{2r_ah_a}+h_b\sqrt{2r_bh_b}+h_c\sqrt{2r_ch_c}\right)} \ \textbf{(32)}$$

From $l_a = AI + \frac{l_a}{h_a}r$ (and analogous) and $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r} \rightarrow \frac{l_a}{h_a}+\frac{l_a}{h_b}+\frac{l_a}{h_c}=\frac{l_a}{r}=\frac{AI}{r}+\frac{l_a}{h_a}$

$$\rightarrow \frac{l_a}{h_b}+\frac{l_a}{h_c}=\frac{AI}{r} \text{ (and analogous)}$$

From $\frac{l_a}{h_b}+\frac{l_a}{h_c}=\frac{AI}{r}$ (and analogous) after summation:

$$\frac{AI+BI+CI}{r}=\sum \frac{l_b+l_c}{h_a} \ \textbf{(33)}$$

From (14) and (33):

$$\sum \sqrt{\frac{r_a}{h_a}} = \sqrt{\frac{2R-r}{r}+\sum \frac{l_b+l_c}{h_a}} \ \textbf{(34)}$$

From (15) and (34):

$$\sum \frac{\sqrt{2Rr}}{n_a-\sqrt{(b-c)^2+r^2}} \geq \sqrt{\frac{2R-r}{r}+\sum \frac{l_b+l_c}{h_a}} \ \textbf{(35)}$$

From (18) and (33):

$$\sum \sqrt{\frac{r_a-r}{r}} = \sqrt{2\left(\frac{2R-r}{r}+\sum \frac{l_b+l_c}{h_a}\right)} \ \textbf{(36)}$$

From (20) and (33):

$$\sum \sqrt{\frac{n_a}{n_a-\sqrt{(b-c)^2+4r^2}}-1} = \sqrt{2\left(\frac{2R-r}{r}+\sum \frac{l_b+l_c}{h_a}\right)} \textbf{(37)}$$

From (24) and (33):

$$\sum \frac{n_a^2}{h_a\left(s-\sqrt{2r_ah_a}\right)} = \frac{s}{r}+\sqrt{2\left(\frac{2R-r}{r}+\sum \frac{l_b+l_c}{h_a}\right)} \ \textbf{(38)}$$

From (38) and (B):

$$\frac{s}{r}+\sqrt{2\left(\frac{2R-r}{r}+\sum \frac{l_b+l_c}{h_a}\right)} \geq \frac{(n_a+n_b+n_c)^2}{s(h_a+h_b+h_c)-\left(h_a\sqrt{2r_ah_a}+h_b\sqrt{2r_bh_b}+h_c\sqrt{2r_ch_c}\right)} \ \textbf{(39)}$$

$\max(r_a,r_b,r_c) \geq r_a$ (and analogous) $\rightarrow \frac{\max(r_a,r_b,r_c)}{h_a} \geq \frac{r_a}{h_a}$ (and analogous) and after summation:

$\max(r_a,r_b,r_c)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right) \geq \frac{r_a}{h_a}+\frac{r_b}{h_b}+\frac{r_c}{h_c}=\frac{2R-r}{r}$ and $\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}=\frac{1}{r}$ we obtain:

$$\frac{\max(r_a,r_b,r_c)}{r} \geq \frac{2R-r}{r} \rightarrow \max(r_a,r_b,r_c) \geq 2R-r \ \textbf{(40)}$$

From (14) and (40):

$$\sqrt{\frac{\max(r_a,r_b,r_c)+AI+BI+CI}{r}} \geq \sum\sqrt{\frac{r_a}{h_a}}\ \textbf{(41)}$$

From (19) and (40):

$$\sqrt{2(\max(r_a,r_b,r_c)+AI+BI+CI)} \geq \sum\sqrt{r_a-r}\ \textbf{(42)}$$

From (20) and (40):

$$\sqrt{\frac{2(\max(r_a,r_b,r_c)+AI+BI+CI)}{r}} \geq \sum\sqrt{\frac{n_a}{n_a-\sqrt{(b-c)^2+4r^2}}-1}\ \textbf{(43)}$$

From (24) and (40):

$$\frac{s}{r}+\sqrt{\frac{2(\max(r_a,r_b,r_c)+AI+BI+CI)}{r}} \geq \sum\frac{n_a^2}{h_a\left(s-\sqrt{2r_ah_a}\right)}\ \textbf{(44)}$$

From (28) and (40):

$$\frac{s}{r}+\sqrt{\frac{2(\max(r_a,r_b,r_c)+AI+BI+CI)}{r}} \geq \frac{(n_a+n_b+n_c)^2}{s(h_a+h_b+h_c)-\left(h_a\sqrt{2r_ah_a}+h_b\sqrt{2r_bh_b}+h_c\sqrt{2r_ch_c}\right)}\ \textbf{(45)}$$

Using same idea from (1) for g_a ,we obtain:

$$\mathbf{AI + r \geq g_a\ \text{(and analogous)(46)}}$$

From (1) and (46):

$$2AI+\sqrt{(b-c)^2+r^2}+r \geq n_a+g_a\ \textbf{(and analogous)(47)}$$

From $AI=\frac{b+c}{2s}l_a$ and (47):

$$2\frac{b+c}{2s}l_a \geq n_a+g_a-\sqrt{(b-c)^2+r^2}-r$$

$$\frac{b+c}{s} \geq \frac{n_a+g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ \textbf{(and analogous)(48)}$$

From (48) after summation:

$$4 \geq \sum\frac{n_a+g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ \textbf{(49)}$$

From $l_a=AI+\frac{l_a}{h_a}r$ (and analogous) and (46):

$$l_a \geq AI+r \geq g_a\ \textbf{(and analogous)(50)}$$

From (1) after summation:

$$AI+BI+CI \geq n_a+n_b+n_c-\sum\sqrt{(b-c)^2+r^2}\ \textbf{(51)}$$

From(51) and

$$AI=\frac{r}{\sin\frac{A}{2}} \rightarrow \sum\frac{1}{\sin\frac{A}{2}} \geq \frac{n_a+n_b+n_c-\sum\sqrt{(b-c)^2+r^2}}{r}\ \textbf{(52)}$$

From(14) and (51):

$$\sum\sqrt{\frac{r_a}{h_a}} \geq \sqrt{\frac{2R-r+n_a+n_b+n_c-\sum\sqrt{(b-c)^2+r^2}}{r}}\ \textbf{(53)}$$

From(19) and (51):

$$\sum\sqrt{r_a-r} \geq \sqrt{2\left(2R-r+n_a+n_b+n_c-\sum\sqrt{(b-c)^2+r^2}\right)}\ \textbf{(54)}$$

From(20) and (51):

$$\sum\sqrt{\frac{n_a}{n_a-\sqrt{(b-c)^2+4r^2}}-1}\ge\sqrt{\frac{2\left(2R-r+n_a+n_b+n_c-\sum\sqrt{(b-c)^2+r^2}\right)}{r}}\ (55)$$

From(24) and (51):

$$\sum\frac{n_a^2}{h_a\left(s-\sqrt{2r_ah_a}\right)}\ge\frac{s}{r}+\sqrt{\frac{2\left(2R-r+n_a+n_b+n_c-\sum\sqrt{(b-c)^2+r^2}\right)}{r}}\ (56)$$

From (33) and (51):

$$\sum\frac{l_b+l_c}{h_a}\ge\frac{n_a+n_b+n_c-\sum\sqrt{(b-c)^2+r^2}}{r}\ (57)$$

From $n_a\ge m_a\frac{l_a}{g_a}$ (and analogous) [6] and (1):

$$m_a\frac{l_a}{g_a}\le AI+\sqrt{(b-c)^2+r^2}\ \text{(and analogous)}(58)$$

From (58) after summation:

$$AI+BI+CI\ge\sum m_a\frac{l_a}{g_a}-\sum\sqrt{(b-c)^2+r^2}\ (59)$$

From (59) and

$$AI=\frac{r}{\sin\frac{A}{2}}:\sum\frac{1}{\sin\frac{A}{2}}\ge\frac{\sum m_a\frac{l_a}{g_a}-\sum\sqrt{(b-c)^2+r^2}}{r}\ (60)$$

From (59) and (14):

$$\sum\sqrt{\frac{r_a}{h_a}}\ge\sqrt{\frac{\sum m_a\frac{l_a}{g_a}-\sum\sqrt{(b-c)^2+r^2}}{r}}\ (61)$$

From (59) and (19):

$$\sum\sqrt{r_a-r}\ge\sqrt{2\left(2R-r+\sum m_a\frac{l_a}{g_a}-\sum\sqrt{(b-c)^2+r^2}\right)}(62)$$

From (20) and (59):

$$\sum\sqrt{\frac{n_a}{n_a-\sqrt{(b-c)^2+4r^2}}-1}\ge\sqrt{\frac{2\left(2R-r+\sum m_a\frac{l_a}{g_a}-\sum\sqrt{(b-c)^2+r^2}\right)}{r}}\ (63)$$

From (21) and (63):

$$2\sum\frac{\sqrt{Rr}}{n_a-\sqrt{(b-c)^2+r^2}}\ge\sqrt{\frac{2\left(2R-r+\sum m_a\frac{l_a}{g_a}-\sum\sqrt{(b-c)^2+r^2}\right)}{r}}\ (64)$$

From (24) and (59):

$$\sum\frac{n_a^2}{h_a\left(s-\sqrt{2r_ah_a}\right)}\ge\frac{s}{r}+\sqrt{\frac{2\left(2R-r+\sum m_a\frac{l_a}{g_a}-\sum\sqrt{(b-c)^2+r^2}\right)}{r}}\ (65)$$

From (33) and (59):

$$\sum\frac{l_b+l_c}{h_a}\geq\frac{\sum m_a\frac{l_a}{g_a}-\sum\sqrt{(b-c)^2+r^2}}{r}\ (66)$$

From (41) and (61):

$$\sqrt{\frac{\max(r_a,r_b,r_c)+AI+BI+CI}{r}}\geq\sqrt{\frac{\sum m_a\frac{l_a}{g_a}-\sum\sqrt{(b-c)^2+r^2}}{r}}\ (67)$$

From (47) and $n_a\geq m_a\frac{l_a}{g_a}$ (and analogous):

$$2AI+\sqrt{(b-c)^2+r^2}+r\geq m_a\frac{l_a}{g_a}+g_a\ (\text{and analogous})(68)$$

From (48) and $n_a\geq m_a\frac{l_a}{g_a}$ (and analogous):

$$\frac{b+c}{s}\geq\frac{m_a\frac{l_a}{g_a}+g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ \text{(and analogous)(69)}$$

From (69) after summation:

$$\frac{2(a+b+c)}{s}\geq\sum\frac{m_a\frac{l_a}{g_a}+g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\rightarrow 4\geq\sum\frac{m_a\frac{l_a}{g_a}+g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ (70)$$

From (69):

$$\frac{b+c}{s}\geq\frac{m_a}{g_a}+\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\rightarrow\prod\left(\frac{b+c}{s}-\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\right)\geq\frac{m_am_bm_c}{g_ag_bg_c}\ (71)$$

From (70):

$$4\geq\sum\frac{m_a}{g_a}+\sum\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a},\ \sum\frac{m_a}{g_a}\geq 3\rightarrow 1\geq\sum\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ (72)$$

From $\frac{b+c}{s}\geq\frac{m_a}{g_a}+\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}$ (and analogous) and $m_a^2\geq r_br_c\sqrt{\frac{p_a}{l_a}}$ (and analogous)[7]:

$$\frac{b+c}{s}\geq\frac{\sqrt{r_br_c}}{g_a}\sqrt[4]{\frac{p_a}{l_a}}+\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ \text{(and analogous) (73)}$$

From (73) after summation:

$$4\geq\sum\frac{\sqrt{r_br_c}}{g_a}\sqrt[4]{\frac{p_a}{l_a}}+\sum\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ (74)$$

From (73) :

$$\prod\left(\frac{b+c}{s}-\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\right)\geq\frac{r_ar_br_c}{g_ag_bg_c}\sqrt[4]{\frac{p_ap_bp_c}{g_ag_bg_c}}\ (75)$$

From (73) and $2\frac{\sqrt{r_br_c}}{l_a}=\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}$ (and analogous):

$$2\frac{(b+c)}{s}\geq\sqrt[4]{\frac{p_a}{l_a}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)+2\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ \text{(and analogous) (76)}$$

From (76) and $p_a\geq l_a$:

$$2\frac{(b+c)}{s}\geq\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}+2\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ \text{(and analogous)(77)}$$

$2\left[\frac{(b+c)}{s}+\frac{(a+c)}{s}+\frac{(b+a)}{s}\right]=4\frac{a+b+c}{s}$ and $a+b+c=2s\rightarrow 2\left[\frac{(b+c)}{s}+\frac{(a+c)}{s}+\frac{(b+a)}{s}\right]=4\frac{2s}{s}=8$

From (76) after summation:

$$8\geq \sum \sqrt[4]{\frac{p_a}{l_a}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)+2\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a} \quad (78)$$

From (77) after summation:

$$8\geq \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}+\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{b}}+\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}+2\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a} \quad (79)$$

From: $p_a \geq m_a$ and (76):

$$2\frac{(b+c)}{s}\geq \sqrt[4]{\frac{m_a}{l_a}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)+2\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a} \text{ (and analogous)}(80)$$

From (80) after summation:

$$8\geq \sum \sqrt[4]{\frac{m_a}{l_a}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)+2\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a} \quad (81)$$

From $p_a \geq \sqrt{r_b r_c}$ and (76): $2\frac{(b+c)}{s}\geq \sqrt[4]{\frac{\sqrt{r_b r_c}}{l_a}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)+2\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}$ (and analogous)

From $2\frac{\sqrt{r_b r_c}}{l_a}=\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}$ (and analogous) $\rightarrow \frac{\sqrt{r_b r_c}}{l_a}=\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)$ and $2\frac{(b+c)}{s}\geq$
$\sqrt[4]{\frac{\sqrt{r_b r_c}}{l_a}}\left(\sqrt{\frac{b}{c}}++\sqrt{\frac{c}{b}}\right)+2\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}$ (and analogous):

$$2\frac{(b+c)}{s}\geq \left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}+2\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a} \text{ (and analogous) } (82)$$

From (82) after summation:

$$8\geq \sum\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)}+2\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a} \quad (83)$$

Now we consider: $\sqrt{(b-c)^2+r^2}+r-g_a \geq n_a-2AI$ (from(47)):

$$\frac{\sqrt{(b-c)^2+r^2}+r-g_a}{l_a}\geq \frac{n_a}{l_a}-\frac{2AI}{l_a}=\frac{n_a}{l_a}-\frac{b+c}{s}$$

$$\frac{b+c}{s}\geq \frac{n_a}{l_a}+\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a} \text{ (and analogous)}(84) \text{ and after summation:}$$

$$\frac{2(a+b+c)}{s}\geq \frac{n_a}{l_a}+\frac{n_b}{l_b}+\frac{n_c}{l_c}+\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a} \rightarrow 4\geq \frac{n_a}{l_a}+\frac{n_b}{l_b}+\frac{n_c}{l_c}+\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a} \quad (85)$$

From (85):

$$7\geq \frac{n_a+l_a}{l_a}+\frac{n_b+l_b}{l_b}+\frac{n_c+l_c}{l_c}+\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a} \quad (86)$$

From (86) and AM-GM inequality:

$$7\geq 2\left(\sqrt{\frac{n_a}{l_a}}+\sqrt{\frac{n_b}{l_b}}+\sqrt{\frac{n_c}{l_c}}\right)+\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a} \quad (87)$$

From $n_a \geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogous) [8] and (85):

$$4\geq \frac{p_a}{\sqrt{l_a g_a}}+\frac{p_b}{\sqrt{l_b g_b}}+\frac{p_c}{\sqrt{l_c g_c}}+\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ (88)$$

From $n_a \geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogous) and (1):

$$p_a\sqrt{\frac{l_a}{g_a}} \leq AI+\sqrt{(b-c)^2+r^2} \text{ (and analogous) (89)}$$

From $n_a \geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogous) and (84):

$$\frac{b+c}{s} \geq \frac{p_a}{\sqrt{l_a g_a}}+\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a} \text{ (and analogous)(90)}$$

From (86) and (50):

$$7\geq \frac{n_a+g_a}{l_a}+\frac{n_b+g_b}{l_b}+\frac{n_c+g_c}{l_c}+\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ (91)$$

From (91) and $n_a+g_a \geq 2m_a$[6] (and analogous):

$$7\geq 2\left(\frac{m_a}{l_a}+\frac{m_b}{l_b}+\frac{m_c}{l_c}\right)+\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ (92)$$

From (92) and $m_a^2 \geq r_b r_c\sqrt{\frac{p_a}{l_a}}$ (and analogous):

$$7\geq 2\sum \frac{\sqrt{r_b r_c}}{l_a}\sqrt[4]{\frac{p_a}{l_a}}+\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ (93)$$

From (93) and $2\frac{\sqrt{r_b r_c}}{l_a}=\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}$ (and analogous): $7\geq \sum \sqrt[4]{\frac{p_a}{l_a}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)+\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ (94)$

From (94) and $p_a \geq m_a$ (and analogous):

$$7\geq \sum \sqrt[4]{\frac{m_a}{l_a}}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)+\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ (95)$$

From (95) and $m_a^2 \geq r_b r_c\sqrt{\frac{p_a}{l_a}}$ (and analogous) and:

$$7\geq \sum \left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\sqrt[4]{\frac{p_a}{l_a}}}+\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ (95)$$

From (95):

$$7 \geq \sum \left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\sqrt[4]{\frac{p_a}{l_a}}}}}+\sum \frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a}\ (96)$$

From (96):

$$7 \geq \sum\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\sqrt[4]{\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)\sqrt[4]{\dots}}}}+\sum\frac{g_a-\sqrt{(b-c)^2+r^2}-r}{l_a} \quad (97)$$

$\frac{h_a}{r}=1+1+\frac{h_a}{r_a}=\frac{2r_a+h_a}{r_a}$ (and analogous)

From $\frac{n_a}{h_a}=\frac{\sqrt{(b-c)^2+4r^2}}{2r}$ (and analogous)[3] $\rightarrow \frac{h_a}{r}=\frac{2n_a}{\sqrt{(b-c)^2+4r^2}}$ and $\frac{h_a}{r}=\frac{2r_a+h_a}{r_a}$ (and analogous) :

$\frac{2n_a}{\sqrt{(b-c)^2+4r^2}}=\frac{2r_a+h_a}{r_a} \rightarrow$

$$\sqrt{(b-c)^2+4r^2}=\frac{2n_ar_a}{2r_a+h_a} \text{ (and analogous)(98)}$$

Is shown that: $\sqrt{(b-a)^2+4r^2}, \sqrt{(b-c)^2+4r^2}, \sqrt{(a-c)^2+4r^2}$ are sides of a triangle[3] and from (98):

$$\frac{n_ar_a}{2r_a+h_a}, \frac{n_br_b}{2r_b+h_b}, \frac{n_cr_c}{2r_c+h_c} \text{ are sides of a triangle(99)}$$

From $s^2=n_a^2+2r_ah_a$ (and analogous):

$2r_ah_a=s^2-n_a^2=(s-n_a)(s+n_a) \rightarrow \frac{2r_ah_a}{s+n_a}=s-n_a; \frac{s}{h_a}=\frac{a}{2r} \rightarrow \frac{a}{2r}=\frac{n_a}{h_a}++\frac{2r_a}{s+n_a}$ (and analogous)

$\frac{a}{2r}=\frac{\sqrt{(b-c)^2+4r^2}}{2r}+\frac{2r_a}{s+n_a}$ and (98):

$$a=\frac{2n_ar_a}{2r_a+h_a}+\frac{4rr_a}{s+n_a} \text{ (and analogous) (100)}$$

From (100):

$$\frac{n_ar_a}{2r_a+h_a}+\frac{2rr_a}{s+n_a}, \frac{n_br_b}{2r_b+h_b}+\frac{2rr_b}{s+n_b}, \frac{n_cr_c}{2r_c+h_c}+\frac{2rr_c}{s+n_c} \text{ are sides of a triangle (101)}$$

From (100):

$$\frac{a}{2r_a}=\frac{n_a}{2r_a+h_a}+\frac{2r}{s+n_a} \text{ (and analogous)(102)}$$

From (102) and abc=4RS and $r_ar_br_c=Ss$:

$$\prod\left(\frac{n_a}{2r_a+h_a}+\frac{2r}{s+n_a}\right)=\frac{R}{2s} \text{ (103)}$$

Now:

$$n_a-\sqrt{(b-c)^2+4r^2}=n_a\left(1-\frac{2r_a}{2r_a+h_a}\right)=\frac{n_ah_a}{2r_a+h_a} \text{ (and analogous)(104)}$$

From $\frac{h_a}{r}=\frac{2r_a+h_a}{r_a}$ (and analogous) and $\frac{r_a}{r}=\frac{n_a}{n_a-\sqrt{(b-c)^2+4r^2}}$ (and analogous)[4]:

$$\frac{r_a}{r}=\frac{n_a}{n_a-\sqrt{(b-c)^2+4r^2}}=1+\frac{2r_a}{h_a} \text{ (and analogous)(105)}$$

If $AA' = n_a$ (and analogous); $N_a -$ Nagel point; $AN_a = \sqrt{(b-c)^2 + 4r^2}$ (and analogous)
$n_a = AN_a + A'N_a$ (and analogous);

$$\mathbf{A'N_a = n_a - \sqrt{(b-c)^2 + 4r^2} = \frac{n_a h_a}{2r_a + h_a}} \textbf{ (and analogous)(from (104)) (106)}$$

$$\mathbf{\frac{AN_a}{A'N_a} = \frac{\frac{2n_a r_a}{2r_a + h_a}}{\frac{n_a h_a}{2r_a + h_a}} = \frac{2r_a}{h_a}} \textbf{ (and analogous)(107)}$$

From (14) and (107):

$$\mathbf{\sum \sqrt{\frac{AN_a}{A'N_a}} = \sqrt{\frac{2(2R - r + AI + BI + CI)}{r}}} \textbf{ (108)}$$

From (107) and $\frac{r_a}{h_a} = \frac{r_a - r}{2r}$ (and analogous):

$$\mathbf{\frac{AN_a}{A'N_a} = \frac{r_a - r}{r}} \textbf{ (and analogous)(109)}$$

From $\frac{a}{2r} = \frac{n_a}{h_a} + \frac{2r_a}{s + n_a}$ (and analogous) and (1):

$$\mathbf{\frac{AI + \sqrt{(b-c)^2 + r^2}}{h_a} + \frac{2r_a}{s + n_a} \geq \frac{a}{2r}} \textbf{ (and analogous)(110)}$$

From (110) after summation:

$$\mathbf{\sum \frac{AI + \sqrt{(b-c)^2 + r^2}}{h_a} + 2\sum \frac{r_a}{s + n_a} \geq \frac{s}{r}} \textbf{ (111)}$$

References:

[1]. CĂTĂLIN BARBU-PUNCTE, DREPTE ŞI CERCURI REMARCABILE ASOCIATE UNUI TRIUNGHI
[2]. Bogdan Fuștei-NEW IDENTITIES AND INEQUALITIES IN TRIANGLE-(II)-www.smrmh.ro
[3]. Bogdan Fuștei-NEW TRIANGLES IN TERMS OF NAGEL'S CEVIANS)-www.smrmh.ro
[4]. Bogdan Fuștei-NAGEL'S CEVIANS REVISITED)-www.smrmh.ro
[5].Bogdan Fuștei-ABOUT NAGEL'S AND GERGONNE'S CEVIANS (II))-www.smrmh.ro
[6].Bogdan Fuştei-ABOUT NAGEL AND GERGONNE CEVIANS (III))-www.smrmh.ro
[7]. Bogdan Fuștei-NEW INEQUALITIES WITH SPIEKER'S CEVIAN)-www.smrmh.ro
[8]. Bogdan Fuștei-CONNECTIONS BETWEEN FAMOUS CEVIANS)-www.smrmh.ro