NEW GEOMETRIC INEQUALITY PROPOSAL

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1 Problem Statement

Let P be any point in the interior region of a triangle ABC with inradius r. Let D, E, and F be the mid-points of the sides BC, AC, and AB, respectively. If $PD = x$, $PE = y$, and $PF = z$, prove that:

$$x + y + z \geq 3r$$

2 Proof and Solution

Step 1: Consider the inner triangle DEF formed by the mid-points of triangle ABC. Let its sides be a, b, c and its area be S_{inner}. By applying Fermat's inequality to this inner triangle DEF with respect to the interior point P, we obtain:

$$(x + y + z)^2 \geq \frac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}S_{\text{inner}}$$

Step 2: For any triangle with sides a, b, c and inradius r_{inner}, the following fundamental geometric inequality holds:

$$a^2 + b^2 + c^2 \geq 36r_{\text{inner}}^2$$

Step 3: Since triangle DEF is formed by the midlines of triangle ABC, its linear dimensions are exactly half of the large triangle ABC. Therefore, its inradius is exactly half of the large triangle's inradius:

$$r_{\text{inner}} = \frac{r}{2}$$

Step 4: Substituting this relation into the fundamental inequality from Step 2 gives:

$$a^2 + b^2 + c^2 \geq 36\left(\frac{r}{2}\right)^2 = 36 \cdot \frac{r^2}{4} = 9r^2$$

Step 5: Now, substituting the result $a^2 + b^2 + c^2 \geq 9r^2$ back into Fermat's inequality (Step 1) yields:

$$(x + y + z)^2 \geq \frac{9r^2}{2} + 2\sqrt{3}S_{\text{inner}}$$

Step 6: Furthermore, by the geometric area inequality for the inner triangle DEF, we know that $S_{\text{inner}} \geq 3\sqrt{3}r_{\text{inner}}^2$. Substituting $r_{\text{inner}} = \frac{r}{2}$ gives:

$$S_{\text{inner}} \geq 3\sqrt{3}\left(\frac{r}{2}\right)^2 = \frac{3\sqrt{3}r^2}{4}$$

Step 7: Replacing S_{inner} in our main expression from Step 5 with this lower bound:

$$(x+y+z)^2 \geq \frac{9r^2}{2} + 2\sqrt{3}\left(\frac{3\sqrt{3}r^2}{4}\right)$$

$$(x+y+z)^2 \geq \frac{18r^2}{4} + \frac{18r^2}{4} = \frac{36r^2}{4} = 9r^2$$

Step 8: Taking the positive square root of both sides gives the desired elegant result:

$$x+y+z \geq 3r$$

Equality Conditions

Equality holds if and only if triangle ABC is equilateral and the interior point P coincides exactly with its center (incenter/circumcenter).