José Luis **Díaz–Barrero**
Barcelona Math Circle (BMC)
jose.luis.diaz@upc.edu

L:1299. *Proposed by Ionel-Tudor Călugăreni, Giurgiu, Romania.* Determine $m \in \mathbb{R}$ and solve the equation $8x^4 + mx^2 + x + 1 = 0$, given that the product of two of its roots is $-\frac{1}{2}$.

Solution by Jose Luis Diaz Barrero, Brcelona, Spain. To solve this problem, we can use Vieta's formulas. Let the four roots of the quartic equation $8x^4 + 0x^3 + mx^2 + x + 1 = 0$ be x_1, x_2, x_3, and x_4. First, we divide the entire equation by 8 to make it monic

$$x^4 + 0x^3 + \frac{m}{8}x^2 + \frac{1}{8}x + \frac{1}{8} = 0$$

According to Vieta's formulas, we have

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1x_2 + (x_1 + x_2)(x_3 + x_4) + x_3x_4 &= \frac{m}{8} \\ x_1x_2(x_3 + x_4) + x_3x_4(x_1 + x_2) &= -\frac{1}{8} \\ x_1x_2x_3x_4 &= \frac{1}{8} \end{aligned}$$

Since the product of two roots, say x_1, x_2, is $-\frac{1}{2}$, then

$$x_1x_2 = -\frac{1}{2}$$

Using that $x_1x_2x_3x_4 = \frac{1}{8}$, we can find the product of the other two roots (x_3x_4):

$$\left(-\frac{1}{2}\right) x_3x_4 = \frac{1}{8} \Rightarrow x_3x_4 = -\frac{1}{4}$$

Now, let us define variables for the sums of these pairs

- Let $S_1 = x_1 + x_2$
- Let $S_2 = x_3 + x_4$

Since $x_1 + x_2 + x_3 + x_4 = 0$, then

$$S_1 + S_2 = 0 \Rightarrow S_2 = -S_1$$

Substitute $x_1x_2 = -\frac{1}{2}$, $x_3x_4 = -\frac{1}{4}$, and $S_2 = -S_1$ into relation $x_1x_2(x_3 + x_4) + x_3x_4(x_1 + x_2) = -\dfrac{1}{8}$, we get

$$\left(-\frac{1}{2}\right) S_2 + \left(-\frac{1}{4}\right) S_1 = -\frac{1}{8}$$

$$\frac{1}{2}S_1 - \frac{1}{4}S_1 = -\frac{1}{8}$$

$$\frac{1}{4}S_1 = -\frac{1}{8} \Rightarrow S_1 = -\frac{1}{2}$$

Since $S_2 = -S_1$, we get $S_2 = \dfrac{1}{2}$.
Now we substitute x_1x_2, x_3x_4, S_1, and S_2 into relation $x_1x_2 + (x_1 + x_2)(x_3 + x_4) + x_3x_4 = \dfrac{m}{8}$, and we have

$$x_1x_2 + S_1S_2 + x_3x_4 = \frac{m}{8}$$

$$-\frac{1}{2} + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(-\frac{1}{4}\right) = \frac{m}{8}$$

$$-\frac{1}{2} - \frac{1}{4} - \frac{1}{4} = \frac{m}{8}$$

$$-1 = \frac{m}{8} \Rightarrow m = -8$$

Since $-8 \in \mathbb{R}$, this is a valid value.

Now we find the actual roots by solving the two quadratic subsystems:

1. Finding x_1 and x_2. These roots have a sum of $S_1 = -\dfrac{1}{2}$ and a product of $P_1 = -\dfrac{1}{2}$. They are the roots of:

$$t^2 - S_1t + P_1 = 0 \Rightarrow t^2 + \frac{1}{2}t - \frac{1}{2} = 0 \Leftrightarrow (2t - 1)(t + 1) = 0$$

Thus, the first two roots are:

$$x_1 = \frac{1}{2}, \quad x_2 = -1$$

2. Finding x_3 and x_4. These roots have a sum of $S_2 = \frac{1}{2}$ and a product of $P_2 = -\frac{1}{4}$. They are the roots of

$$t^2 - S_2 t + P_2 = 0 \Rightarrow t^2 - \frac{1}{2}t - \frac{1}{4} = 0 \Leftrightarrow 4t^2 - 2t - 1 = 0$$

Applying the quadratic formula, we obtain

$$t = \frac{1 \pm \sqrt{5}}{4}$$

Thus, the remaining two roots are:

$$x_3 = \frac{1 + \sqrt{5}}{4}, \quad x_4 = \frac{1 - \sqrt{5}}{4}$$

Finally, we conclude that

- Parameter value: $m = -8$
- Equation's roots: $x \in \left\{ -1, \frac{1}{2}, \frac{1 - \sqrt{5}}{4}, \frac{1 + \sqrt{5}}{4} \right\}$