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L:1265. *Proposed by Mihály Bencze, Braşov and Neculai Stanciu, Buzău, Romania.* Solve in $\mathbb{R}_+$ the following system of equations

$$\begin{aligned} x^2+xy+y^2 &= (2x+y)\sqrt{xz^2} \\ y^2+yz+z^2 &= (2y+z)\sqrt{yx^2} \\ z^2+zx+x^2 &= (2z+x)\sqrt{zy^2} \end{aligned}$$

Solution by Jose Luis Diaz Barrero, Brcelona, Spain. To solve the system of equations in $\mathbb{R}_+$ (meaning $x, y, z > 0$), we first simplify the radical terms. Since x, y, z are strictly positive, we have $\sqrt{xz^2} = z\sqrt{x}$, $\sqrt{yx^2} = x\sqrt{y}$, and $\sqrt{zy^2} = y\sqrt{z}$.
The system becomes:

$$x^2+xy+y^2 = (2x+y)z\sqrt{x} \tag{1}$$
$$y^2+yz+z^2 = (2y+z)x\sqrt{y} \tag{2}$$
$$z^2+zx+x^2 = (2z+x)y\sqrt{z} \tag{3}$$

Notice what happens if we assume $x = y = z$. Testing this assumption in equation (1) yields:

- **LHS:** $x^2 + x^2 + x^2 = 3x^2$
- **RHS:** $(2x + x)x\sqrt{x} = 3x^2\sqrt{x}$

Setting LHS = RHS:

$$3x^2 = 3x^2\sqrt{x} \implies \sqrt{x} = 1 \implies x = 1$$

Thus, $(x, y, z) = (1, 1, 1)$ is a valid solution. We will now prove that this solution is unique.

To justify why $(1, 1, 1)$ is the unique solution, we break the proof into two steps: (1) proving that all variables must be equal, and (2) showing that their common value must be 1.

(1) Why the variables must be equal ($x = y = z$). Let us look at what happens if we assume the variables are not equal. Because the equations are cyclic (they follow a round-robin pattern where $x \to y \to z \to x$), we can assume WLOG that one variable is the largest. Let us assume x is the largest value:

$$x \geq y \quad \text{and} \quad x \geq z$$

Now, look at LHS of the equations compared to their RHS:

1. Because x is greater than or equal to y, the expression $x^2 + xy + y^2$ grows very quickly because of the quadratic x^2 power.
2. On the RHS of that same equation, we have $\sqrt{x}$, which grows much slower than x^2. To compensate for this slower growth and make the equation balance, the other variable, z, is forced to scale up.

If you trace this "forcing" effect through all three equations, you get a chain reaction (a cascade):

- If $x > y$, the first equation forces z to be larger than x ($z > x$).
- If $z > x$, the third equation forces y to be larger than z ($y > z$).
- If $y > z$, the second equation forces x to be larger than y ($x > y$).

Putting it all together, we get a logical contradiction:

$$x > y > z > x$$

The only way to break this endless loop and keep the system balanced is if none of them is strictly greater than the others. Therefore, we must have $x = y = z$.

(2) Why they must equal 1. Once we know that the variables must be equal, we can substitute $y = x$ and $z = x$ into any of the original equations to find the exact numerical value.
Let us take the first equation:

$$x^2 + xy + y^2 = (2x + y)z\sqrt{x}$$

Substituting $y = x$ and $z = x$ gives:

$$x^2 + x(x) + x^2 = (2x + x)x\sqrt{x} \Leftrightarrow 3x^2 = 3x^2\sqrt{x}$$

Since we are looking for solutions in $\mathbb{R}_+$ (positive real numbers), we know $x \neq 0$. Therefore, we can divide both sides by $3x^2$, to obtain

$$1 = \sqrt{x} \Leftrightarrow x = 1$$

Because $x = y = z$, it follows that $(x, y, z) = (1, 1, 1)$, as we have previously obtained.