José Luis **Díaz–Barrero**
Barcelona Math Circle (BMC)
jose.luis.diaz@upc.edu

L:1264. *Proposed by Gobej Cristina-Dumitra, Curtea de Argeş, Romania.* Let $a_1, a_2, \ldots, a_n$ be non negative real numbers such that $a_1^2 + a_2^2 + \ldots + a_n^2 = S$. Prove that for all integer $n \geq 2$, it holds

$$\sqrt{S - a_1^2} + \sqrt{S - a_2^2} + \ldots + \sqrt{S - a_n^2} \geq \sqrt{n-1}\,(a_1 + a_2 + \ldots + a_n).$$

Solution by Jose Luis Diaz Barrero, Brcelona, Spain. We will prove this inequality by squaring both sides and applying the Cauchy-Schwarz inequality. Indeed, we want to show that

$$\left(\sum_{i=1}^{n} \sqrt{S - a_i^2}\right)^2 \geq (n-1)\left(\sum_{i=1}^{n} a_i\right)^2$$

Expanding the LHS, we get

$$\text{LHS} = \sum_{i=1}^{n} (S - a_i^2) + 2 \sum_{1 \leq i < j \leq n} \sqrt{(S - a_i^2)(S - a_j^2)}$$

Since $\displaystyle\sum_{i=1}^{n} a_i^2 = S$, the first term simplifies to

$$\sum_{i=1}^{n} (S - a_i^2) = nS - \sum_{i=1}^{n} a_i^2 = nS - S = (n-1)S,$$

and the LHS becomes

$$\text{LHS} = (n-1)S + 2 \sum_{1 \leq i < j \leq n} \sqrt{(S - a_i^2)(S - a_j^2)}$$

Next we apply CBS inequality. Note that we can express the terms inside the radicals as sums of squares

$$S - a_i^2 = \sum_{k \neq i} a_k^2 \quad \text{and} \quad S - a_j^2 = \sum_{k \neq j} a_k^2$$

By the Cauchy-Schwarz inequality:

$$\left(\sum_{k\neq i} a_k^2\right)\left(\sum_{k\neq j} a_k^2\right) \geq \left(\sum_{k\neq i,j} a_k^2 + a_j a_i\right)^2$$

Taking the square root of both sides gives a lower bound for each radical term

$$\sqrt{(S-a_i^2)(S-a_j^2)} \geq \sum_{k\neq i,j} a_k^2 + a_i a_j$$

Substituting this lower bound back into our expanded LHS equation yields:

$$\text{LHS} \geq (n-1)S + 2\sum_{1\leq i<j\leq n}\left(\sum_{k\neq i,j} a_k^2 + a_i a_j\right)$$

Next, we proceed to count the occurrences of each term in the double summation:

1. Each product term $a_i a_j$ (for $i < j$) appears exactly once, giving $2\sum_{i<j} a_i a_j$.
2. Each square term a_k^2 appears in the inner sum whenever $k \neq i$ and $k \neq j$. For a fixed k, the number of ways to choose pairs (i, j) such that neither index equals k is $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$.

Multiplying by the factor of 2 outside the summation, the total coefficient for each a_k^2 is $(n-1)(n-2)$. Since $\sum a_k^2 = S$, the double sum simplifies to

$$(n-1)(n-2)S + 2\sum_{1\leq i<j\leq n} a_i a_j$$

Combining this back into the complete LHS expression, yields

$$\text{LHS} \geq (n-1)S + (n-1)(n-2)S + 2\sum_{1\leq i<j\leq n} a_i a_j$$

$$\text{LHS} \geq (n-1)^2 S + 2\sum_{1\leq i<j\leq n} a_i a_j$$

Now, let us expand the RHS of our target squared inequality

$$\text{RHS} = (n-1)\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(S + 2\sum_{1\le i<j\le n} a_i a_j\right)$$

$$\text{RHS} = (n-1)S + 2(n-1)\sum_{1\le i<j\le n} a_i a_j$$

Subtracting the RHS from our lower bound of the LHS:

$$\text{LHS}-\text{RHS} \geq \left[(n-1)^2 S + 2\sum_{i<j} a_i a_j\right] - \left[(n-1)S + 2(n-1)\sum_{i<j} a_i a_j\right]$$

$$\text{LHS} - \text{RHS} \geq (n-1)(n-2)S - 2(n-2)\sum_{1\le i<j\le n} a_i a_j$$

$$\text{LHS} - \text{RHS} \geq (n-2)\left[(n-1)S - 2\sum_{1\le i<j\le n} a_i a_j\right]$$

Using the algebraic identity

$$(n-1)\sum a_i^2 - 2\sum_{i<j} a_i a_j = \sum_{i<j}(a_i - a_j)^2,$$

we can rewrite the terms inside the brackets:

$$\text{LHS} - \text{RHS} \geq (n-2)\sum_{1\le i<j\le n}(a_i - a_j)^2$$

Since $n \geq 2$, we have $(n-2) \geq 0$, and the sum of squared differences is always non-negative. Therefore,

$$\text{LHS} \geq \text{RHS}$$

Taking the square root of both sides concludes the proof:

$$\sqrt{S-a_1^2} + \sqrt{S-a_2^2} + \ldots + \sqrt{S-a_n^2} \geq \sqrt{n-1}\,(a_1 + a_2 + \ldots + a_n)$$

Finally, we analyze the equality case:

- For $n = 2$, the factor $(n-2)$ becomes 0, meaning equality holds for all non-negative a_1, a_2.
- For $n > 2$, equality holds if and only if $a_1 = a_2 = \ldots = a_n = \sqrt{\frac{S}{n}}$.