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L:1263. *Proposed by Mihály Bencze, Braşov and Neculai Stanciu, Buzău, Romania.* If $x, y, z > 0$, $n \in \mathbb{N}$ and

$$(xy)^{n-1} + (yz)^{n-1} + (zx)^{n-1} = 1,$$

then prove that $x^{2n+1} + y^{2n+1} + z^{2n+1} \geq xyz.$

Solution by Jose Luis Diaz–Barrero, Brcelona, Spain. We can use the given condition to rewrite the right-hand side of the inequality, and then apply the Weighted AM-GM Inequality.

On account of the given constrain $(xy)^{n-1}+(yz)^{n-1}+(zx)^{n-1} = 1$, we have

$$xyz \cdot 1 = xyz \left((xy)^{n-1} + (yz)^{n-1} + (zx)^{n-1}\right)$$

Expanding this expression gives:

$$xyz(x^{n-1}y^{n-1} + y^{n-1}z^{n-1} + z^{n-1}x^{n-1}) = x^n y^n z + xy^n z^n + x^n yz^n$$

Thus, the inequality we need to prove simplifies to:

$$x^{2n+1} + y^{2n+1} + z^{2n+1} \geq x^n y^n z + xy^n z^n + x^n yz^n$$

We can generate each term on the right-hand side by choosing appropriate weights for the terms on the left-hand side. By the Weighted AM-GM inequality, using weights that sum to 1:

1. For the term $x^n y^n z$, assign weights $\frac{n}{2n+1}$, $\frac{n}{2n+1}$, and $\frac{1}{2n+1}$ to x^{2n+1}, y^{2n+1}, and z^{2n+1} respectively:

$$\frac{n}{2n+1}x^{2n+1} + \frac{n}{2n+1}y^{2n+1} + \frac{1}{2n+1}z^{2n+1}$$

$$\geq \left(x^{2n+1}\right)^{\frac{n}{2n+1}} \left(y^{2n+1}\right)^{\frac{n}{2n+1}} \left(z^{2n+1}\right)^{\frac{1}{2n+1}} = x^n y^n z$$

2. Similarly, by cyclic permutation for the term $xy^n z^n$:

$$\frac{1}{2n+1}x^{2n+1} + \frac{n}{2n+1}y^{2n+1} + \frac{n}{2n+1}z^{2n+1} \geq xy^n z^n$$

3. And for the term $x^n y z^n$:

$$\frac{n}{2n+1}x^{2n+1} + \frac{1}{2n+1}y^{2n+1} + \frac{n}{2n+1}z^{2n+1} \geq x^n y z^n$$

Adding these three inequalities together yields

$$\left(\frac{n+1+n}{2n+1}\right)x^{2n+1} + \left(\frac{n+n+1}{2n+1}\right)y^{2n+1} + \left(\frac{1+n+n}{2n+1}\right)z^{2n+1}$$
$$\geq x^n y^n z + x y^n z^n + x^n y z^n,$$

which simplifies precisely to

$$x^{2n+1} + y^{2n+1} + z^{2n+1} \geq x^n y^n z + x y^n z^n + x^n y z^n$$

As established previously, the right-hand side is identically equal to xyz. Therefore,

$$x^{2n+1} + y^{2n+1} + z^{2n+1} \geq xyz.$$

To find the exact conditions for equality, we look at the conditions under which the Weighted AM-GM inequalities applied before hold as equalities. For any AM-GM inequality, equality holds if and only if all the variables being averaged are equal. In this case, the terms being averaged are x^{2n+1}, y^{2n+1}, and z^{2n+1}. Therefore, we must have:

$$x^{2n+1} = y^{2n+1} = z^{2n+1}$$

Since $x, y, z > 0$ and the exponent $2n+1$ is strictly positive, this simplifies directly to $x = y = z$. Now, we substitute $x = y = z$ back into the original constraint equation $(xy)^{n-1} + (yz)^{n-1} + (zx)^{n-1} = 1$, and we obtain

$$(x \cdot x)^{n-1} + (x \cdot x)^{n-1} + (x \cdot x)^{n-1} = 1 \Leftrightarrow 3x^{2n-2} = 1$$

Isolating x^{2n-2}, and taking the $(2n-2)$-th root on both sides (assuming $n > 1$), we find:

$$x = \left(\frac{1}{3}\right)^{\frac{1}{2n-2}} = 3^{-\frac{1}{2n-2}}$$

Thus, the equality case holds if and only if

$$x = y = z = \frac{1}{\sqrt[2n-2]{3}}.$$

This completes the proof.

Note. Observe that if $n = 1$ the constrain does not hold.