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L:1301. *Proposed by Năstăşelu Amelia Curcă, Filiaşi, Mirela Ţacu, Craiova, Romania.* Let $P \in \mathbb{R}[X]$ be a polynomial of degree $n \in \mathbb{N}^*$. If

$$P(X) = X^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \cdots + a_1 X + a_0$$

and $a_{n-1}+a_{n-2}+\cdots+a_1+a_0 \in [-2,0]$, prove that the polynomial P has at least one zero $x_k \in \mathbb{C}$ such that $|x_k - 1| \leq 1$.

Solution by Jose Luis Diaz Barrero, Barcelona, Spain. To prove that the polynomial $P(X)$ has at least one root $x_k \in \mathbb{C}$ such that $|x_k - 1| \leq 1$, we can use the relationship between the zeros of a polynomial and its coefficients. Let $x_1, x_2, \ldots, x_n \in \mathbb{C}$ be the zeros of the polynomial $P(X)$. Since $P(X)$ is a monic polynomial of degree n, we can write it in its factored form over the complex numbers as

$$P(X) = (X - x_1)(X - x_2)\cdots(X - x_n)$$

First, let us evaluate $P(1)$ using the given expanded form of the polynomial

$$P(1) = 1^n + a_{n-1}(1)^{n-1} + a_{n-2}(1)^{n-2} + \cdots + a_1(1) + a_0$$

$$P(1) = 1 + (a_{n-1} + a_{n-2} + \cdots + a_1 + a_0)$$

We are given that the sum of the coefficients lies in the interval $[-2, 0]$. That is,

$$a_{n-1} + a_{n-2} + \cdots + a_1 + a_0 \in [-2, 0]$$

Adding 1 to all parts of this interval, we get

$$P(1) \in [1-2, 1+0] \Rightarrow P(1) \in [-1, 1]$$

Thus, we have the inequality

$$|P(1)| \leq 1$$

Now, we evaluate $P(1)$ using the factored form of the polynomial, to obtain

$$P(1) = (1 - x_1)(1 - x_2) \cdots (1 - x_n)$$

Taking the absolute value on both sides, we obtain

$$|P(1)| = |1 - x_1| \cdot |1 - x_2| \cdots |1 - x_n| = |x_1 - 1| \cdot |x_2 - 1| \cdots |x_n - 1|$$

Suppose, for sake of contradiction, that none of the zeros x_k satisfy $|x_k - 1| \leq 1$. This means that for every zero x_i (where $i = 1, 2, \ldots, n$), we must have

$$|x_i - 1| > 1$$

If we multiply these n strictly positive inequalities together, we get

$$|x_1 - 1| \cdot |x_2 - 1| \cdots |x_n - 1| > 1^n$$

which simplifies to $|P(1)| > 1$. This directly contradicts our previous result, which established that $|P(1)| \leq 1$. Therefore, our initial assumption must be false. There must exist at least one zero $x_k \in \mathbb{C}$ such that

$$|x_k - 1| \leq 1$$