José Luis **Díaz–Barrero**
Barcelona Math Circle (BMC)
jose.luis.diaz@upc.edu

L:1300. *Proposed by Mihaela Cioplea, Daniela Beldea , Băileşti, Craiova, Romania.* Let $P \in \mathbb{R}[X]$ be a polynomial of degree $n \in \mathbb{N}^*$ with all real zeros. If

$$P(X) = X^n + a_1 X^{n-1} + a_2 X^{n-2} + \cdots + a_{n-1} X + a_n$$

and $a_1 + a_2 + \cdots + a_n \in [-2, 0]$, show that the polynomial P has at least one zero in the interval $[0, 2]$.

Solution by Jose Luis Diaz Barrero, Barcelona, Spain. Evaluating the polynomial at $X = 1$, we have

$$P(1) = 1 + (a_1 + a_2 + \cdots + a_n)$$

We are given that the sum of the coefficients satisfies

$$-2 \leq a_1 + a_2 + \cdots + a_n \leq 0$$

Adding 1 to all sides of the inequality gives us

$$-1 \leq 1 + a_1 + a_2 + \cdots + a_n \leq 1$$

This means that

$$|P(1)| \leq 1$$

Since $P(X)$ is a monic polynomial of degree n with all real zeros, we can write it in its factored form. If $x_1, x_2, \ldots, x_n \in \mathbb{R}$ are the zeros of $P(X)$, then

$$P(X) = (X - x_1)(X - x_2) \ldots (X - x_n)$$

Evaluating this factored form at $X = 1$, we get

$$P(1) = (1 - x_1)(1 - x_2) \ldots (1 - x_n)$$

Taking the absolute value of both sides, we obtain

$$|P(1)| = |1 - x_1| \cdot |1 - x_2| \ldots |1 - x_n| \leq 1$$

Assume for the sake of contradiction that $P(X)$ has no zeros in the interval $[0, 2]$. This means that for every zero x_i, either $x_i < 0$ or $x_i > 2$. We distinguish these two cases:

- Case 1: If $x_i < 0$, then $-x_i > 0$, which means $1 - x_i > 1$. Therefore, $|1 - x_i| > 1$.
- Case 2: If $x_i > 2$, then $x_i - 1 > 1$, which means $|1 - x_i| > 1$.

Since every single zero must fall into one of these two cases, we have

$$|1 - x_i| > 1 \quad \text{for all } i = 1, 2, \ldots, n$$

Multiplying these n strictly greater-than-one inequalities together yields

$$|1 - x_1| \cdot |1 - x_2| \ldots |1 - x_n| > 1$$

and

$$|P(1)| > 1$$

We have reached a contradiction. Previously, we proved that $|P(1)| \leq 1$, but our assumption showed that $|P(1)| > 1$. Therefore, our assumption must be false. The polynomial $P(X)$ must have at least one zero in the interval $[0, 2]$.