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L:1298. *Proposed by Molea Gheorghe, Curtea de Argeş, Romania.* Prove that all real roots of the equation

$$x^{6n} - 4x^{2n} - 4 = 0$$

have an absolute value greater than $\sqrt[2n]{2}$, for all $n \in \mathbb{N}^*$.

Solution by Jose Luis Diaz Barrero, Barcelona, Spain. Let $x \in \mathbb{R}$ be a real root of the equation $x^{6n} - 4x^{2n} - 4 = 0$. To analyze the equation, we use the substitution $t = x^{2n}$. Since x is a real number and $2n$ is an even integer (as $n \in \mathbb{N}^*$), we must have $t \geq 0$. Substituting $t = x^{2n}$ into the original equation gives

$$(x^{2n})^3 - 4(x^{2n}) - 4 = 0 \Rightarrow t^3 - 4t - 4 = 0$$

We want to show that any real root x satisfies $|x| > \sqrt[2n]{2}$. In terms of our substitution $t = x^{2n}$, this is equivalent to proving that $t > 2$.

Let us define the function $f : [0, \infty) \to \mathbb{R}$ by

$$f(t) = t^3 - 4t - 4$$

We evaluate $f(t)$ at a few key points

- For $t = 2$ is $f(2) = -4 < 0$
- For $t = 3$ is $f(3) = 11 > 0$

Since $f(2) < 0$ and $f(3) > 0$, by the Intermediate Value Theorem, there must be a real root t_0 in the interval $(2, 3)$. To understand the behavior of $f(t)$ for $t \geq 0$, we find its derivative:

$$f'(t) = 3t^2 - 4$$

Setting $f'(t) = 0$ yields $t = \frac{2}{\sqrt{3}}$.

- For $t \in \left[0, \frac{2}{\sqrt{3}}\right)$, we have $f'(t) < 0$, so the function is strictly decreasing.

- For $t > \frac{2}{\sqrt{3}}$, we have $f'(t) > 0$, so the function is strictly increasing.

Since $f(t)$ is strictly increasing for all $t \geq \frac{2}{\sqrt{3}}$, it can cross the t-axis at most once in this region. Thus, the root $t_0 \in (2, 3)$ is the unique positive real root of the equation. Since t_0 is the only positive real root and $t_0 > 2$, any valid substitution value t must satisfy $t > 2$. Recalling $t = x^{2n}$, we have

$$x^{2n} > 2 \Leftrightarrow |x| > \sqrt[2n]{2}$$

Thus, all real roots of the equation have an absolute value strictly greater than $\sqrt[2n]{2}$ for all $n \in \mathbb{N}^*$, as we wanted to prove.