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L:1297. *Proposed by Vasile Roxana, Udrea Cristian, Craiova, Romania.* Determine the natural numbers n such that

$$\frac{\varphi(n)}{n} = \frac{16}{35},$$

where $\varphi(n)$ represents the number of positive integers less than or equal to n that are coprime to n (Euler's totient function).

Solution by Jose Luis Diaz Barrero, Barcelona, Spain. To find all natural numbers n such that $\frac{\varphi(n)}{n} = \frac{16}{35}$, we can use the product formula for Euler's totient function. Thus, for any natural number n with prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, the ratio $\frac{\varphi(n)}{n}$ is given by

$$\frac{\varphi(n)}{n} = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right),$$

where $p_1, p_2, \ldots, p_k$ are the distinct prime factors of n.

We are given

$$\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right) = \frac{16}{35}$$

We can rewrite each term $\left(1 - \frac{1}{p_i}\right)$ as $\frac{p_i - 1}{p_i}$. This simplifies to

$$\frac{(p_1 - 1)(p_2 - 1)\cdots(p_k - 1)}{p_1 p_2 \cdots p_k} = \frac{16}{35}$$

The fraction on the right-hand side has a denominator of 35. The prime factorization of 35 is $35 = 5 \times 7$. Since the denominator on the left-hand side is the product of the distinct prime factors of n (possibly reduced if there are common factors in the numerator), the primes 5 and 7 must be prime factors of n. Let us check if

n can have only the prime factors 5 and 7. If the distinct prime factors of n are exactly $\{5, 7\}$, then

$$\frac{\varphi(n)}{n} = \left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right) = \frac{4}{5} \times \frac{6}{7} = \frac{24}{35}$$

Since $\frac{24}{35} \neq \frac{16}{35}$, n must have at least one other prime factor. Let the set of distinct prime factors of n be $\{5, 7, q_1, q_2, \ldots, q_m\}$. The equation becomes

$$\frac{24}{35} \times \left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_m}\right) = \frac{16}{35}$$

Divide both sides by $\dfrac{24}{35}$, to obtain

$$\left(1 - \frac{1}{q_1}\right) \cdots \left(1 - \frac{1}{q_m}\right) = \frac{16}{35} \times \frac{35}{24} = \frac{16}{24} = \frac{2}{3}$$

We need the product of terms of the form $\dfrac{q_i - 1}{q_i}$ to equal $\dfrac{2}{3}$. Looking at the denominator of $\dfrac{2}{3}$, we see a 3. This means 3 must be one of our primes. Let us test if $q_1 = 3$ works by itself:

$$1 - \frac{1}{3} = \frac{2}{3}$$

This is an exact match. Therefore, the set of distinct prime factors of n must be exactly

$$\{3, 5, 7\}$$

Since the formula for $\dfrac{\varphi(n)}{n}$ only depends on the distinct prime factors of n and not their exponents, any number n whose prime factorization contains only the primes $3, 5,$ and 7 (each raised to any positive integer power) will satisfy the equation. Thus, the natural numbers n searched are of the form

$$n = 3^a \cdot 5^b \cdot 7^c,$$

where a, b, c are positive integers.