José Luis **Díaz–Barrero**
Barcelona Math Circle (BMC)
jose.luis.diaz@upc.edu

L:1266. *Proposed by Gheoreghe Ghiţă, Buzău, Romania.* Let $x, y, z > 0$ such that $xyz = 1$. Prove that

$$\frac{9}{x^3+y^3+z^3+3} \leq \frac{x}{x^2+yz} + \frac{y}{y^2+zx} + \frac{z}{z^2+xy} \leq \frac{x^2+y^2+z^2}{2}.$$

Solution by Jose Luis Diaz Barrero, Brcelona, Spain. We will split the proof into two parts: verifying the RHS inequality first, followed by the LHS inequality.

1. Proving the Right-Hand Side (RHS) Inequality. We want to show that

$$\frac{x}{x^2+yz} + \frac{y}{y^2+zx} + \frac{z}{z^2+xy} \leq \frac{x^2+y^2+z^2}{2}$$

Using AM-GM inequality, we can bound the denominator of each term. Notice that for the first term, we have

$$x^2 + yz \geq 2\sqrt{x^2 \cdot yz} = 2x\sqrt{yz}$$

Using the condition $xyz = 1$, we can substitute $\sqrt{yz} = \frac{1}{\sqrt{x}}$, which yields

$$x^2 + yz \geq 2x \cdot \frac{1}{\sqrt{x}} = 2\sqrt{x}$$

Thus, we obtain the bound:

$$\frac{x}{x^2+yz} \leq \frac{x}{2\sqrt{x}} = \frac{\sqrt{x}}{2}$$

Summing this cyclically for all three terms gives:

$$\sum_{\text{cyc}} \frac{x}{x^2+yz} \leq \frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{2}$$

By the Cauchy-Schwarz inequality (or standard power mean inequalities), we know that:

$$\sqrt{x} + \sqrt{y} + \sqrt{z} \leq x + y + z \leq x^2 + y^2 + z^2$$

for all $x, y, z > 0$ with $xyz = 1$. Therefore, we have:

$$\sum_{\text{cyc}} \frac{x}{x^2 + yz} \leq \frac{\sqrt{x} + \sqrt{y} + \sqrt{z}}{2} \leq \frac{x^2 + y^2 + z^2}{2}$$

This completes the proof of the RHS inequality.

2. Proving the Left-Hand Side (LHS) Inequality. We want to show that

$$\frac{9}{x^3 + y^3 + z^3 + 3} \leq \frac{x}{x^2 + yz} + \frac{y}{y^2 + zx} + \frac{z}{z^2 + xy}$$

First, we multiply the numerator and denominator of each fraction by its corresponding variable (x, y, and z respectively), to get

$$\frac{x}{x^2 + yz} = \frac{x^2}{x^3 + xyz}$$

Since $xyz = 1$, the expression simplifies to

$$\sum_{\text{cyc}} \frac{x}{x^2 + yz} = \frac{x^2}{x^3 + 1} + \frac{y^2}{y^3 + 1} + \frac{z^2}{z^3 + 1}$$

Now, we apply the CBS inequality, to obtain

$$\frac{x^2}{x^3 + 1} + \frac{y^2}{y^3 + 1} + \frac{z^2}{z^3 + 1} \geq \frac{(x + y + z)^2}{(x^3 + 1) + (y^3 + 1) + (z^3 + 1)}$$
$$= \frac{(x + y + z)^2}{x^3 + y^3 + z^3 + 3}$$

Applying AM-GM to x, y, z, we have

$$\frac{x + y + z}{3} \geq \sqrt[3]{xyz} = \sqrt[3]{1} = 1 \Rightarrow x + y + z \geq 3$$

Squaring both sides gives $(x+y+z)^2 \geq 9$. Substituting this back into our Cauchy-Schwarz result yields:

$$\sum_{\text{cyc}} \frac{x}{x^2 + yz} \geq \frac{(x + y + z)^2}{x^3 + y^3 + z^3 + 3} \geq \frac{9}{x^3 + y^3 + z^3 + 3}$$

This completes the proof of the LHS inequality. Since both inequalities have been proven independently, it holds

$$\frac{9}{x^3 + y^3 + z^3 + 3} \leq \frac{x}{x^2 + yz} + \frac{y}{y^2 + zx} + \frac{z}{z^2 + xy} \leq \frac{x^2 + y^2 + z^2}{2}$$

with equality if and only if $x = y = z = 1$.