ROMANIAN MATHEMATICAL MAGAZINE

JP.615 In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{h_c^2}+\frac{b^4+c^4}{h_a^2}+\frac{c^4+a^4}{h_b^2}\geq 96r^2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\frac{a^4+b^4}{h_c^2}+\frac{b^4+c^4}{h_a^2}+\frac{c^4+a^4}{h_b^2}=\sum_{cyc}\frac{a^4+b^4}{h_c^2}\geq$$

$$\overset{AM-GM}{\geq}\sum_{cyc}\frac{2a^2b^2}{\frac{4F^2}{c^2}}=\frac{1}{2F^2}\sum_{cyc}a^2b^2c^2=\frac{3}{2F^2}\cdot(4RF)^2=\frac{3}{2F^2}\cdot 16R^2F^2=$$

$$=24R^2\overset{EULER}{\geq}24\cdot(2r)^2=24\cdot 4r^2=96r^2$$

Equality holds for $a=b=c$.

Solution 2 by Marin Chirciu– Romania

$$\sum\frac{b^4+c^4}{h_a^2}=\sum\frac{b^4}{h_a^2}+\sum\frac{c^4}{h_a^2}\overset{Holder}{\geq}\frac{\left(\sum b\right)^4}{9\sum h_a^2}+\frac{\left(\sum c\right)^4}{9\sum h_a^2}=\frac{(2p)^4+(2p)^4}{9\sum h_a^2}=\frac{32p^4}{9\sum h_a^2}\leq\frac{32p^4}{9\cdot p^2}=$$

$$=\frac{32p^2}{9}\overset{Mitinovic}{\geq}\frac{32\left(3r\sqrt{3}\right)^2}{9}=\frac{32\cdot 27\cdot r^2}{9}=96r^2=RHS.$$

Equality holds for $a=b=c$.

Solution 3 by Omkumar Rao-India

$$\textit{The altitudes are}:$$

$$h_a=\frac{2S}{a},h_b=\frac{2S}{b},h_c=\frac{2S}{c},\qquad LHS=\frac{c^2(a^4+b^4)+a^2(b^4+c^4)+b^2(c^4+a^4)}{4S^2}$$

$$By\ AM-GM, x^4+y^4\geq 2x^2y^2\ for\ all\ real\ x,y.$$

$$LHS\geq\frac{6a^2b^2c^2}{4S^2},\qquad By\ extended\ law\ of\ sines: abc=4RS$$

$$Euler's\ Inequality: R\geq 2r$$

$$LHS\geq 96r^2$$

Equality holds if and only if both the AM – GM Inequality and Euler's Inequality are tight, which occurs if and only if ΔABC is equilateral.