ROMANIAN MATHEMATICAL MAGAZINE

JP.615 In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{h_c^2}+\frac{b^4+c^4}{h_a^2}+\frac{c^4+a^4}{h_b^2}\ge 96r^2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\frac{a^4+b^4}{h_c^2}+\frac{b^4+c^4}{h_a^2}+\frac{c^4+a^4}{h_b^2}=\sum_{cyc}\frac{a^4+b^4}{h_c^2}\ge$$

$$\overset{AM-GM}{\ge}\sum_{cyc}\frac{2a^2b^2}{\frac{4F^2}{c^2}}=\frac{1}{2F^2}\sum_{cyc}a^2b^2c^2=\frac{3}{2F^2}\cdot(4RF)^2=\frac{3}{2F^2}\cdot 16R^2F^2=$$

$$=24R^2\overset{EULER}{\ge}24\cdot(2r)^2=24\cdot 4r^2=96r^2$$

Equality holds for $a=b=c$.

Solution 2 by Marin Chirciu– Romania

$$\sum\frac{b^4+c^4}{h_a^2}=\sum\frac{b^4}{h_a^2}+\sum\frac{c^4}{h_a^2}\overset{Holder}{\ge}\frac{\left(\sum b\right)^4}{9\sum h_a^2}+\frac{\left(\sum c\right)^4}{9\sum h_a^2}=\frac{(2p)^4+(2p)^4}{9\sum h_a^2}=\frac{32p^4}{9\sum h_a^2}\le\frac{32p^4}{9\cdot p^2}=$$

$$=\frac{32p^2}{9}\overset{Mitinovic}{\ge}\frac{32\left(3r\sqrt{3}\right)^2}{9}=\frac{32\cdot 27\cdot r^2}{9}=96r^2=RHS.$$

Equality holds for $a=b=c$.