ROMANIAN MATHEMATICAL MAGAZINE

JP.614 In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{h_c}+\frac{b^4+c^4}{h_a}+\frac{c^4+a^4}{h_b}\geq 288r^3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\frac{a^4+b^4}{h_c}+\frac{b^4+c^4}{h_a}+\frac{c^4+a^4}{h_b}=\sum_{cyc}\frac{a^4+b^4}{h_c}\geq$$

$$\overset{AM-GM}{\geq}\sum_{cyc}\frac{2a^2b^2}{h_c}=\sum_{cyc}\frac{2a^2b^2}{\frac{2F}{c}}=\frac{1}{F}\sum_{cyc}a^2b^2c=$$

$$=\frac{abc}{F}\sum_{cyc}ab=\frac{4RF}{F}(s^2+r^2+4Rr)\geq 4R\left(\frac{27R^2}{4}+r^2+4Rr\right)\overset{EULER}{\geq}$$

$$\geq 4\cdot 2r\left(\frac{27}{4}\cdot 4r^2+r^2+4\cdot 2r\cdot r\right)=8r(27R^2+9r^2)=8r\cdot 36r^2=288r^3$$

Equality holds for $a=b=c$.

Solution 2 by Marin Chirciu – Romania

$$\sum\frac{b^4+c^4}{h_a}=\sum\frac{b^4}{h_a}+\sum\frac{c^4}{h_a}\overset{Holder}{\geq}\frac{\left(\sum b\right)^4}{9\sum h_a}+\frac{\left(\sum c\right)^4}{9\sum h_a}=\frac{(2p)^4+(2p)^4}{9\sum h_a}=\frac{32p^4}{9\sum h_a}\overset{Santalo}{\geq}\frac{32p^4}{9\cdot p\sqrt{3}}=$$

$$=\frac{32p^3}{9\sqrt{3}}\overset{Mitinovic}{\geq}\frac{32\left(3r\sqrt{3}\right)^3}{9\sqrt{3}}=\frac{32\cdot 27\cdot 3\sqrt{3}r^3}{9\sqrt{3}}=288r^3=RHS\ .$$

Equality holds for $a=b=c$.

Solution 3 by Omkumar Rao-India

$$The\ altitudes\ are\ given\ by: h_c=\frac{2S}{c}, h_a=\frac{2S}{a}, h_b=\frac{2S}{b}$$
$$: S-Area\ of\ \Delta ABC, s-semiperimeter, r-inradius, R-circumradius$$

$$LHS=\frac{c(a^4+b^4)}{2S}+\frac{a(b^4+c^4)}{2S}+\frac{b(c^4+a^4)}{2S}$$

$$By\ AM-GM, x^4+y^4\geq 2x^2y^2$$
$$for\ all\ real\ x,y.$$

$$LHS \geq \frac{a.2b^2c^2 + b.2c^2a^2 + c.2a^2b^2}{2S}$$

$$LHS \geq \frac{2abc(ab+bc+ca)}{2S}$$

$$By\ extended\ law\ of\ sines\ abc = 4RS$$

$$LHS \geq 4R(ab+bc+ca)$$

$$Euler's\ Inequality: R \geq 2r$$

$$ab+bc+ca = s^2 + 4Rr + r^2$$

$$s \geq 3\sqrt{3}r$$

$$LHS \geq 4.2r.36r^2 = 288r^3$$

$$Equality\ holds\ when\ both\ Euler's\ Inequality\ and\ the\ side - inradius\ inequality\ are\ tight, which\ occurs\ if\ and\ only\ if\ \Delta ABC\ is\ equilateral.$$