ROMANIAN MATHEMATICAL MAGAZINE

JP.614 In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{h_c}+\frac{b^4+c^4}{h_a}+\frac{c^4+a^4}{h_b}\ge 288r^3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\frac{a^4+b^4}{h_c}+\frac{b^4+c^4}{h_a}+\frac{c^4+a^4}{h_b}=\sum_{cyc}\frac{a^4+b^4}{h_c}\ge$$

$$\overset{AM-GM}{\ge}\sum_{cyc}\frac{2a^2b^2}{h_c}=\sum_{cyc}\frac{2a^2b^2}{\frac{2F}{c}}=\frac{1}{F}\sum_{cyc}a^2b^2c=$$

$$=\frac{abc}{F}\sum_{cyc}ab=\frac{4RF}{F}(s^2+r^2+4Rr)\ge 4R\left(\frac{27R^2}{4}+r^2+4Rr\right)\overset{EULER}{\ge}$$

$$\ge 4\cdot 2r\left(\frac{27}{4}\cdot 4r^2+r^2+4\cdot 2r\cdot r\right)=8r(27R^2+9r^2)=8r\cdot 36r^2=288r^3$$

Equality holds for $a=b=c$.

Solution 2 by Marin Chirciu – Romania

$$\sum\frac{b^4+c^4}{h_a}=\sum\frac{b^4}{h_a}+\sum\frac{c^4}{h_a}\overset{Holder}{\ge}\frac{\left(\sum b\right)^4}{9\sum h_a}+\frac{\left(\sum c\right)^4}{9\sum h_a}=\frac{(2p)^4+(2p)^4}{9\sum h_a}=\frac{32p^4}{9\sum h_a}\overset{Santalo}{\ge}\frac{32p^4}{9\cdot p\sqrt{3}}=$$

$$=\frac{32p^3}{9\sqrt{3}}\overset{Mitinovic}{\ge}\frac{32\left(3r\sqrt{3}\right)^3}{9\sqrt{3}}=\frac{32\cdot 27\cdot 3\sqrt{3}r^3}{9\sqrt{3}}=288r^3=RHS\,.$$

Equality holds for $a=b=c$.