

ROMANIAN MATHEMATICAL MAGAZINE

JP.610 Solve for real numbers:

$$\log_{2\sqrt{8+2\sqrt{15}}}(x^2 + x + 2) = \log_{\sqrt{4+\sqrt{15}}}(x^2 + x + 1)$$

Proposed by Marian Ursărescu, Florică Anastase – Romania

Solution by proposers

$$\text{Let be } \log_{2\sqrt{8+2\sqrt{15}}}(x^2 + x + 2) = \log_{\sqrt{4+\sqrt{15}}}(x^2 + x + 1) = y \Rightarrow$$

$$\begin{cases} x^2 + x + 1 = \left(\sqrt{4 + \sqrt{15}}\right)^y \\ x^2 + x + 2 = \left(2\sqrt{8 + 2\sqrt{15}}\right)^y \end{cases} \Leftrightarrow (4 + \sqrt{15})^y + 1 = \left(2\sqrt{2} \cdot \sqrt{4 + \sqrt{15}}\right)^y.$$

We observe that $y = 2$ is a solution, $(4 + \sqrt{15})^2 + 1 = (2\sqrt{2} \cdot \sqrt{4 + \sqrt{15}})^2 \Leftrightarrow$

$32 + 8\sqrt{15} = 32 + 8\sqrt{15}$ true. Let us prove that $y = 2$ is the unique solution:

$$1 + \left(\frac{1}{4 + \sqrt{15}}\right)^y = \left(\frac{2\sqrt{2}}{\sqrt{4 + \sqrt{15}}}\right)^y$$

Let be the function $f(y) = 1 + \left(\frac{1}{4 + \sqrt{15}}\right)^y$ strictly decreases because $\frac{1}{4 + \sqrt{15}} < 1$,

and $g(y) = \left(\frac{2\sqrt{2}}{\sqrt{4 + \sqrt{15}}}\right)^y$ strictly increases function because $\frac{2\sqrt{2}}{\sqrt{4 + \sqrt{15}}} > 1$.

Hence, $y = 2$ is the unique solution i.e. $x^2 + x + 1 = 4 + \sqrt{15} \Leftrightarrow x^2 + x - 3 - \sqrt{15} = 0$

$$\text{with } x_{1,2} = \frac{-1 \pm \sqrt{3 + 5\sqrt{15}}}{2}$$