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JP.609 In acute triangle ABC, A', B', C' are symmetric points of the points A, B, C to the sides BC, AC, and AB respectively. Prove that:

$$\frac{\sigma[A'B'C']}{\sigma[ABC]} = 4\left(\frac{r}{R}\right)^2 + 8\cdot\frac{r}{R} - 1$$

where $\sigma[ABC]$ represent area of ΔABC.

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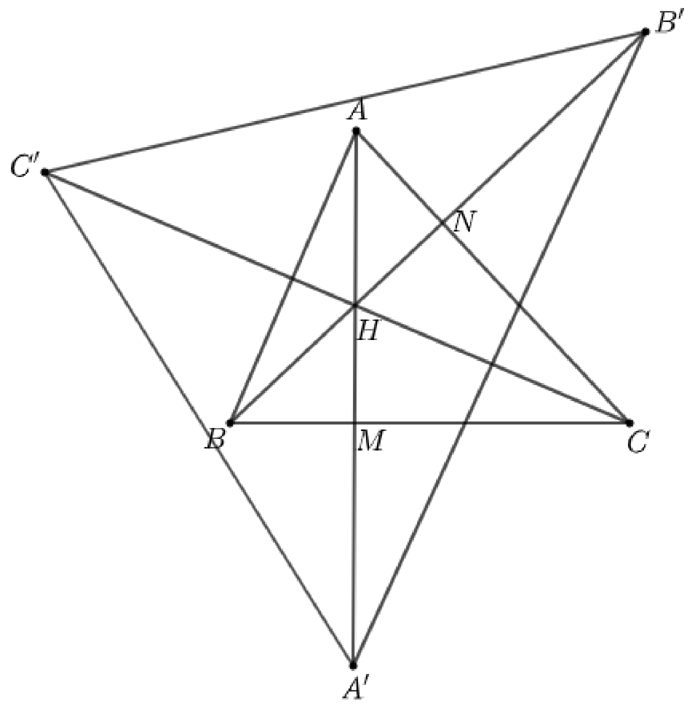
Solution by proposers

Let be $\{H\} = AA' \cap BB' \cap CC'$. From $h_a = 2R\sin B\sin C$ and $HM = 2R\cos B\cos C$,



then $HA' = 2R\cos(B - C)$ and analogs. Hence, we get:

$$\sigma[A'B'C'] = \frac{1}{2}(HA'\cdot HB'\cdot\sin C + HB'\cdot HC'\cdot\sin A + HC'\cdot HA'\cdot\sin B) =$$

$$= 2R^2\sum_{cyc}\cos(B-C)\cos(C-A)\sin C =$$

$$= 2R^2\sin A\sin B\sin C\sum_{cyc}\frac{\cos(B-C)\cos(C-A)}{\sin A\sin B} \quad (1)$$

But $\sigma[ABC] = 2R^2\sin A\sin B\sin C$ and

$$\sum_{cyc}\frac{\cos(B-C)\cos(C-A)}{\sin A\sin B}=3+8\cos A\cos B\cos C\quad (2)$$

From (1) and (2), it follows: $\frac{\sigma[A'B'C']}{\sigma[ABC]}=3+8\cos A\cos B\cos C\quad (3)$

But $\cos A\cos B\cos C=\frac{s^2-(2R+r)^2}{4R^2}$ (4)

From (3) and (4), it follows:

$$\frac{\sigma[A'B'C']}{\sigma[ABC]}=3+\frac{2(s^2-4R^2-4Rr-r^2)}{R^2}\quad (5)$$

Using Walker's inequality $s^2\ge 2R^2+8Rr+3r^2$ to obtain:

$$\frac{\sigma[A'B'C']}{\sigma[ABC]}\ge 3-\frac{4R^2+8Rr+4r^2}{R^2}=4\left(\frac{r}{R}\right)^2+8\cdot\frac{r}{R}-1$$