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JP.608 Let be the triangle ABC, AA_1, BB_1, CC_1 internal bisectors and A_2, B_2, C_2 contact points to the bisectors with the circumcircle of the triangle. Prove that: $A_1A_2 \cdot B_2C_2 + B_1B_2 \cdot A_2C_2 + C_1C_2 \cdot A_2B_2 \geq Rs$ where p represent the semiperimeter and R the circumradii of triangle ABC.

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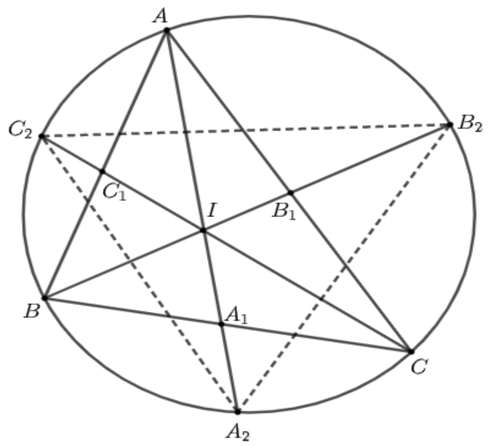
Solution by proposers

We have $\rho(A_1) = AA_1 \cdot A_1A_2 = BA_1 \cdot A_1C \Rightarrow A_1A_2 = \frac{BA_1 \cdot A_1C}{A_1A_2}$ (1)



From Bisector Theorem, we get: $BA_1 = \frac{ac}{b+c}$ and $CA_1 = \frac{ab}{b+c}$ (2)

But $AA_1 = \frac{2bc}{b+c}\cos\frac{A}{2}$ (3)

From (1), (2) and (3), it follows that: $A_1A_2 = \frac{a^2}{2(b+c)\cos\frac{A}{2}}$ (4)

From Law of sines: $\frac{B_2C_2}{\sin\left(\frac{B+C}{2}\right)} = 2R \Rightarrow B_2C_2 = 2R \cdot \cos\frac{A}{2}$ (5)

From (4) and (5) we get: $AA_2 \cdot B_2C_2 = \frac{a^2R}{b+c}$

Therefore, by adding we obtain:

$$\sum_{cyc} A_1A_2 \cdot B_2C_2 = R\sum_{cyc}\frac{a^2}{b+c} \overset{Bergstrom}{\geq} R \cdot \frac{(a+b+c)^2}{2(a+b+c)} = R \cdot \frac{a+b+c}{2} = Rs.$$