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JP.607. *Proposed by Marin Chirciu, Romania.* Solve for reals

$$\sqrt[4]{8x-7}+\sqrt[4]{3-2x^4}=2.$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. For the fourth roots to be defined over the real numbers, the radicands must be non-negative:

1. $8x-7\geq 0 \Rightarrow x\geq \frac{7}{8}$
2. $3-2x^4\geq 0 \Rightarrow x^4\leq \frac{3}{2} \Rightarrow x\leq \sqrt[4]{3/2}$

Thus, any real solution x must lie within the interval:

$$x\in\left[\frac{7}{8},\sqrt[4]{\frac{3}{2}}\right]$$

By inspection, we test $x=1$, which lies within our valid domain:

$$\sqrt[4]{8(1)-7}+\sqrt[4]{3-2(1)^4}=\sqrt[4]{1}+\sqrt[4]{1}=1+1=2$$

Thus, $x=1$ is a valid real solution.

Let us define the function $f(x)=\sqrt[4]{8x-7}+\sqrt[4]{3-2x^4}$ on the interval $\left[\frac{7}{8},\sqrt[4]{\frac{3}{2}}\right]$. To determine its behavior, we compute its first derivative $f'(x)$:

$$f'(x)=\frac{1}{4}(8x-7)^{-3/4}\cdot 8+\frac{1}{4}(3-2x^4)^{-3/4}\cdot(-8x^3)$$

$$f'(x)=\frac{2}{(8x-7)^{3/4}}-\frac{2x^3}{(3-2x^4)^{3/4}}$$

Setting $f'(x)=0$ to find the critical points:

$$\frac{2}{(8x-7)^{3/4}}=\frac{2x^3}{(3-2x^4)^{3/4}}\Rightarrow\frac{1}{(8x-7)^{3/4}}=\frac{x^3}{(3-2x^4)^{3/4}}$$

Raising both sides to the power of $\frac{4}{3}$ yields:

$$\frac{1}{8x-7}=\frac{x^4}{3-2x^4}\Leftrightarrow 8x^5-5x^4-3=0$$

Since we know $x=1$ satisfies this equation, we can factor out $(x-1)$ using polynomial long division or synthetic division:

$$(x-1)(8x^4+3x^3+3x^2+3x+3)=0$$

For any x in our domain $\left[\frac{7}{8},\sqrt[4]{\frac{3}{2}}\right]$, the value of x is strictly positive. Consequently, the polynomial factor $(8x^4+3x^3+3x^2+3x+3)$ is always strictly greater than zero and has no real roots. Therefore, $x=1$ is the unique critical point of $f(x)$ in this domain. We can check the sign of $f'(x)$ around $x=1$:

- For $\frac{7}{8} \leq x < 1$, we have $f'(x) > 0$, meaning $f(x)$ is strictly increasing.
- For $1 < x \leq \sqrt[4]{1.5}$, we have $f'(x) < 0$, meaning $f(x)$ is strictly decreasing.

Since $f(x)$ strictly increases to a peak at $x = 1$ and strictly decreases thereafter, $x = 1$ is a strict global maximum. The maximum value achieved by the function is exactly 2. Thus, the equation $f(x) = 2$ has exactly one real solution $x = 1$.