ROMANIAN MATHEMATICAL MAGAZINE

JP.607 Solve for reals:

$$\sqrt[4]{8x-7}+\sqrt[4]{3-2x^4}=2$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Existence conditions: $8x-7\geq 0$ **and** $3-2x^4\geq 0$.

With means inequality we obtain:

1. $(8x-7)+1+1+1\geq 4\sqrt[4]{(8x-7)\cdot 1\cdot 1\cdot 1}=4\sqrt[4]{8x-7}$, **with equal for**

$8x-7=1\Leftrightarrow x=1.$

2. $(3-2x^4)+1+1+1\geq 4\sqrt[4]{(3-2x^4)\cdot 1\cdot 1\cdot 1}=4\sqrt[4]{3-2x^4}$, **with equal for**

$3-2x^4=1\Leftrightarrow x=1.$

It follows:

$$2=\sqrt[4]{8x-7}+\sqrt[4]{3-2x^4}\leq\frac{(8x-7)+1+1+1}{4}+\frac{(3-2x^4)+1+1+1}{4}=$$

$$=\frac{8x-2x^4+2}{4}\Rightarrow 2\leq\frac{8x-2x^4+2}{4}\Leftrightarrow x^4-4x+3\leq 0\Leftrightarrow$$

$$\Leftrightarrow (x-1)^2(x^2+2x+3)\leq 0\Leftrightarrow x=1.$$

We deduce that $x=1$ **is the unique solution of the equation.**

Solution 2 by Jose Luis Diaz – Barrero – Spain

For the fourth roots to be defined over the real numbers, the radicands must be non-negative:

1. $8x-7\geq 0\Rightarrow x\geq\frac{7}{8}$

2. $3-2x^4\geq 0\Rightarrow x^4\leq\frac{3}{2}\Rightarrow x\leq\sqrt[4]{\frac{3}{2}}$

Thus, any real solution x **must lie within the interval:**

$$x\in\left[\frac{7}{8},\sqrt[4]{\frac{3}{2}}\right]$$

By inspection, we test $x=1$**, which lies within our valid domain:**

$$\sqrt[4]{8(1)-7}+\sqrt[4]{3-2(1)^4}=\sqrt[4]{1}+\sqrt[4]{1}=1+1=2$$

Thus, $x=1$ is a valid real solution.

Let us define the function $f(x)=\sqrt[4]{8x-7}+\sqrt[4]{3-2x^4}$ on the interval $\left[\frac{7}{8},\sqrt[4]{\frac{3}{2}}\right]$.

To determine its behavior, we compute its first derivative $f'(x)$:

$$f'(x)=\frac{1}{4}(8x-7)^{-\frac{3}{4}}\cdot 8+\frac{1}{4}(3-2x^4)^{-\frac{3}{4}}\cdot(-8x^3)$$

$$f'(x)=\frac{2}{(8x-7)^{\frac{3}{4}}}-\frac{2x^3}{(3-2x^4)^{\frac{3}{4}}}$$

Setting $f'(x)=0$ to find the critical points:

$$\frac{2}{(8x-7)^{\frac{3}{4}}}=\frac{2x^3}{(3-2x^4)^{\frac{3}{4}}}\Rightarrow\frac{1}{(8x-7)^{\frac{3}{4}}}=\frac{x^3}{(3-2x^4)^{\frac{3}{4}}}$$

Raising both sides to the power of $\frac{4}{3}$ yields:

$$\frac{1}{8x-7}=\frac{x^4}{3-2x^4}\Leftrightarrow 8x^5-5x^4-3=0$$

Since we know $x=1$ satisfies this equation, we can factor out $(x-1)$ using polynomial long division or synthetic division:

$$(x-1)(8x^4+3x^3+3x^2+3x+3)=0$$

For any x in our domain $\left[\frac{7}{8},\sqrt[4]{\frac{3}{2}}\right]$, the value of x is strictly positive. Consequently, the polynomial factor $(8x^4+3x^3+3x^2+3x+3)$ is always strictly greater than zero and has no real roots. Therefore, $x=1$ is the unique critical point of $f(x)$ in this domain. We can check the sign of $f'(x)$ around $x=1$:

- For $\frac{7}{8}\le x<1$, we have $f'(x)>0$, meaning $f(x)$ is strictly increasing.
- For $1<x\le\sqrt[4]{1.5}$, we have $f'(x)<0$, meaning $f(x)$ is strictly decreasing.

Since $f(x)$ strictly increases to a peak at $x=1$ and strictly decreases thereafter, $x=1$, is a strict global maximum. The maximum value achieved by the function is exactly 2. Thus, the equation $f(x)=2$ has exactly one real solution $x=1$.