ROMANIAN MATHEMATICAL MAGAZINE

JP.606 Solve the following equation:

$$2016^{2-x^2}+2016^{2017x^2-4\sqrt{x}-2013}=2017$$

Proposed by Adrian Gobej – Romania

Solution by proposer

Conditions $x\in[0;+\infty)$. LHS can be written:

$$E(x)=2016\cdot 2016^{1-x^2}+2016^{2017x^2-4\sqrt{x}-2013}=$$

$$=\underbrace{2016^{1-x^2}+2016^{1-x^2}+\cdots+2016^{1-x^2}}_{2016\ terms}+\underbrace{2016^{2017x^2-4\sqrt{x}-2013}}_{1\ term}\overset{AM\geq GM}{\geq}$$

$$\overset{AM\geq GM}{\geq}2017\sqrt[2017]{\underbrace{2016^{1-x^2}\cdot 2016^{1-x^2}\cdot\ldots\cdot 2016^{1-x^2}}_{2016\ terms}\cdot 2016^{2017x^2-4\sqrt{x}-2013}}=$$

$$=2017\sqrt[2017]{2016^{2016-2016x^2}\cdot 2016^{2017x^2-4\sqrt{x}-2013}}=$$

$$=2017\sqrt[2017]{2016^{2016-2016x^2+2017x^2-4\sqrt{x}-2013}}=$$

$$=2017\sqrt[2017]{2016^{x^2-4\sqrt{x}+3}}\ (R_1)$$

As $x^2-4\sqrt{x}+3\underset{not\ \sqrt{x}=t}{=}t^4-4t+3=(t-1)^2(t^2+2t-3)=$ (returning to the notation)$=\left(\sqrt{x}-1\right)^2\left(x+2\sqrt{x}-3\right)$ we obtain using (R_1):

$$\left.\begin{array}{r}E(x)\geq 2017\sqrt[2017]{2016^{\left(\sqrt{x}-1\right)^2\left(x+2\sqrt{x}+3\right)}}\\ \left.\begin{array}{r}As\ \left(\sqrt{x}-1\right)^2\geq 0\\ x+2\sqrt{x}+3=\left(\sqrt{x}+1\right)^2+2\geq 2\end{array}\right\}\Rightarrow\left(\sqrt{x}-1\right)^2\left(x+2\sqrt{x}+3\right)\geq 0\end{array}\right\}\Rightarrow$$

$$\Rightarrow E(x)\geq 2017\sqrt[2017]{2016^0}=2017.$$

As in the previous inequalities we have equality if and only if $x=1\Rightarrow x=1$ is a unique solution of the given equation.