ROMANIAN MATHEMATICAL MAGAZINE

JP.605 Find $x, y \in \mathbb{R}$ such that:

$$4^{x+y} + \sqrt{2026^x + 2026^y - 2} = 2^{x+y+1} - 1.$$

Proposed by Adrian Gobej – Romania

Solution 1 by proposer

The relationship in the statement is written $\sqrt{2026^x + 2026^y - 2} + (2^{x+y} - 1)^2 = 0 \Rightarrow$

$$\Rightarrow \begin{cases} \sqrt{2026^x + 2026^y - 2} = 0 (R_1) \\ \quad\quad\quad and \\ 2^{x+y} - 1 = 0 \Rightarrow x = -y \end{cases}$$

From $x = -y \Rightarrow$ relationship (R_1) becomes $2026^x + 2026^{-x} = 2 \Leftrightarrow$

$\Leftrightarrow 2026^x + \frac{1}{2026^x} = 2 \Leftrightarrow (2026^x - 1)^2 = 0 \Leftrightarrow 2026^x = 1 \Leftrightarrow x = 0$ and so $x = y = 0$.

Solution 2 by Omkumar Rao-India

$$4^{x+y} - 2^{x+y+1} + 1 \quad + \sqrt{2026^x + 2026^y - 2} = 0$$

$$\underbrace{(2^{x+y} - 1)^2}_{\geq 0} + \underbrace{\sqrt{2026^x + 2026^y - 2}}_{\geq 0} = 0$$

$$(2^{x+y} - 1)^2 = 0 \Rightarrow 2^{x+y} = 2^0 \Longrightarrow y = -x$$

$$\sqrt{2026^x + 2026^y - 2} = 0 : (y = -x)$$

$2026^x + \frac{1}{2026^x} = 2$. By AM-GM Inequality, for m>0: $m + \frac{1}{m} \geq 2$

Hence: $2026^x = 1 \Longrightarrow x = 0$ and since x+y=0 , y=0. Therefore,the unique solution is $(x, y) = (0, 0)$.