ROMANIAN MATHEMATICAL MAGAZINE

JP.604 If $a,b>0;n\in\mathbb{N}^*$ then:

$$\frac{a^{2n}}{b^{2n}}+a^{2n}b^{4n}+\frac{1}{a^{4n}b^{2n}}\geq\frac{b}{a}+a^2b+\frac{1}{ab^2}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

For $n=1$ we will prove that:

$$\frac{a^2}{b^2}+a^2b^4+\frac{1}{a^4b^2}\geq\frac{b}{a}+a^2b+\frac{1}{ab^2}\qquad (1)$$

Denote:

$$x=\frac{a}{b};y=ab^2;z=\frac{1}{a^2b}$$

Let's observe that $xyz=1$. Inequality (1) can be written:

$$x^2+y^2+z^2\geq\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\quad x^2+y^2+z^2\geq\frac{xy+yz+zx}{xyz}$$

$$x^2+y^2+z^2\geq xy+yz+zx$$

$$2x^2+2y^2+2z^2\geq 2xy+2yz+2zx$$

$$x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2\geq 0$$

$$(x-y)^2+(y-z)^2+(z-x)^2\geq 0$$

Back to the problem:

$$\frac{a^{2n}}{b^{2n}}+a^{2n}b^{4n}+\frac{1}{a^{4n}b^{2n}}=\frac{\left(\frac{a^2}{b^2}\right)^n}{1^{n-1}}+\frac{(a^2b^4)^n}{2^{n-1}}+\frac{\left(\frac{1}{a^4b^2}\right)^n}{1^{n-1}}\overset{RADON}{\geq}$$

$$\geq\frac{\left(\frac{a^2}{b^2}+a^2b^4+\frac{1}{a^4b^2}\right)^n}{(1+1+1)^{n-1}}=\frac{\left(\frac{a^2}{b^2}+a^2b^4+\frac{1}{a^4b^2}\right)^{n-1}}{3^{n-1}}\cdot\left(\frac{a^2}{b^2}+a^2b^4+\frac{1}{a^4b^2}\right)\geq$$

$$\overset{AM-GM}{\geq}\frac{\left(3\sqrt[3]{\frac{a^2}{b^2}\cdot a^2b^4\cdot\frac{1}{a^4b^2}}\right)^{n-1}}{3^{n-1}}\cdot\left(\frac{a^2}{b^2}+a^2b^4+\frac{1}{a^4b^2}\right)\geq$$

$$\overset{(1)}{\geq}\frac{3^{n-1}}{3^{n-1}}\cdot\left(\frac{b}{a}+a^2b+\frac{1}{ab^2}\right)=\frac{b}{a}+a^2b+\frac{1}{ab^2}$$

Equality holds for $a=b=1$.

Solution 2 by Tapas Das-India

$$Let\ x=\frac{a^{2n}}{b^{2n}},y=a^{2n}b^{4n},z=\frac{1}{a^{4n}b^{2n}}\ then\ \frac{b}{a}=x^{\frac{1}{2n}},a^2b=y^{\frac{1}{2n}},\frac{1}{ab^2}=z^{\frac{1}{2n}},xyz=1$$

$$so\ we\ need\ to\ show, x+y+z\ge x^r+y^r+z^r, where\ r=\frac{1}{2n}<1$$

$$\frac{x+y+z}{3}\overset{AM-GM}{\ge}\sqrt[3]{xyz}\overset{xyz=1}{=}1\ and\ \frac{x^r+y^r+z^r}{3}\le\left(\frac{x+y+z}{3}\right)^r\ as\ r\in(0,1)$$

$$so\ clearly, \frac{x^r+y^r+z^r}{3}\le\left(\frac{x+y+z}{3}\right)^r\le\frac{x+y+z}{3}\left(as\ \frac{x+y+z}{3}\ge 1\ \&\ r\in(0,1)\right)$$

$$so,\quad x+y+z\ge x^r+y^r+z^r$$

Equality holds for x=y=z=1 or a=b=1

Solution 3 by Marin Chirciu-Romania

Using substitutions $(x,y,z)=\left(\frac{a}{b},ab^2,\frac{1}{a^2b}\right)$**,** $x,y,z>0$**,** $xyz=1$ **the inequality can be written:**

$$x^{2n}+y^{2n}+z^{2n}\ge\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Leftrightarrow x^{2n}+y^{2n}+z^{2n}\ge xy+yz+zx \text{ see:}$$

$$x^{2n}+y^{2n}+z^{2n}\overset{SOS}{\ge}x^ny^n+y^nz^n+z^nx^n\overset{Lema}{\ge}xy+yz+zx.$$

<u>Lemma.</u>

If $A,B,C>0$**,** $ABC=1$ **and** $n\in\mathbb{N}^*$ **then**

$$A^n+B^n+C^n\ge A+B+C.$$

<u>Proof:</u>

For $n=1$ **we obtain:** $A+B+C=A+B+C$.

For $n\ge 2$:

$$\sum A^n\overset{Holder}{\ge}\frac{\left(\sum A\right)^n}{3^{n-1}}=\sum A\left(\frac{\sum A}{3}\right)^{n-1}\overset{AM-GM}{\ge}\sum A\left(\sqrt[3]{ABC}\right)^{n-1}=\sum A.$$

Lemma is used for $(A,B,C)=(xy,yz,zx)$**,** $xyz=1\Rightarrow ABC=1$.

Equality holds for $x=y=z\Leftrightarrow\frac{a}{b}=ab^2=\frac{1}{a^2b}\Leftrightarrow a=b=1$.