ROMANIAN MATHEMATICAL MAGAZINE

JP.603 If $a, b, c > 0$ then:

$$2\left(a^a + b^b + c^c\right) \geq a^b + a^c + b^a + b^c + c^a + c^b$$

Proposed by Daniel Sitaru – Romania

Solution 1 by proposer

Lemma: If $a, b > 0$ then:

$$a^a + b^b \geq a^b + b^a \quad (1)$$

Proof:

WLOG: $a \geq b \Rightarrow \ln a \geq \ln b$

$(a, b); (\ln a, \ln b)$ **are similarly ordered.**

By rearrangement's inequality:

$$a \ln a + b \ln b \geq a \ln b + b \ln a$$

Let be $f: \mathbb{R} \to \mathbb{R}; f(x) = e^x$

$f'(x) = e^x; f''(x) = e^x > 0 \Rightarrow f$ **convexe**

By Karamata's inequality:

$$e^{a \ln a} + e^{b \ln b} \geq e^{a \ln b} + e^{b \ln a}$$
$$a^a + b^b \geq a^b + b^a$$

Back to the problem. By (1):

$$a^a + b^b \geq a^b + b^a$$
$$b^b + c^c \geq b^c + c^b \quad (2)$$
$$c^c + a^a \geq a^c + c^a \quad (3)$$

By adding (1); (2); (3):

$$2\left(a^a + b^b + c^c\right) \geq a^b + a^c + b^a + b^c + b^a + c^b + c^a$$

Equality holds for $a = b = c$.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\forall\, x > 0, x^x \overset{?}{\geq} x \Leftrightarrow x.\ln x \overset{?}{\geq} \ln x \Leftrightarrow (\ln x)(x-1) \overset{?}{\geq} 0 \to \text{true}\ \forall\, x \in (0,1)$$
$$and\ also\ \text{true}\ \forall\, x \geq 1 \because \ln x \geq 0\ and\ x - 1 \geq 0 \therefore x^x \geq x\ \forall\, x > 0 \to 1$$
$$\text{We prove}: a^a + \mathrm{b}^\mathrm{b} \geq a^\mathrm{b} + \mathrm{b}^a\ \forall\, a, \mathrm{b} > 0\ and\ \text{if}\ a = \mathrm{b} = 1, \text{then}: \mathrm{LHS} - \mathrm{RHS}$$
$$= 0\ and\ \text{if}\ exactly\ \text{one}\ variable = 1\ and\ \text{WLOG we}\ may\ assume\ a = 1, \text{then}$$
$$\text{remains to prove}: \mathrm{b}^\mathrm{b} \overset{?}{\geq} \mathrm{b} \to \text{true}\ via\ 1; \text{we now dive into the}\ following\ cases:$$
$$\boxed{\text{Case 1}}\ a \geq \mathrm{b} > 1\ or\ b > 1 > a > 0\ and\ \text{then}: a^a + \mathrm{b}^\mathrm{b} \overset{?}{\geq} a^\mathrm{b} + \mathrm{b}^a$$

$$\Leftrightarrow \mathrm{b}^{a-\mathrm{b}}.\left(\left(\frac{a}{\mathrm{b}}\right)^{a}-1\right)\boxed{\underset{(*)}{\overset{?}{\geq}}}\left(\frac{a}{\mathrm{b}}\right)^{\mathrm{b}}-1;\text{Now, }(a-\mathrm{b})\text{ and }(\ln a-\ln \mathrm{b})\text{ have same sign}$$

$$\therefore (a-\mathrm{b})\left(\ln\frac{a}{\mathrm{b}}\right)\geq 0\Rightarrow a\left(\ln\frac{a}{\mathrm{b}}\right)\geq \mathrm{b}\left(\ln\frac{a}{\mathrm{b}}\right)\Rightarrow\left(\frac{a}{\mathrm{b}}\right)^{a}\geq\left(\frac{a}{\mathrm{b}}\right)^{\mathrm{b}}$$

$$(\because \ln\text{ is an }\uparrow\text{ function})\Rightarrow \mathrm{b}^{a-\mathrm{b}}.\left(\left(\frac{a}{\mathrm{b}}\right)^{a}-1\right)\geq \mathrm{b}^{a-\mathrm{b}}.\left(\left(\frac{a}{\mathrm{b}}\right)^{\mathrm{b}}-1\right)\overset{?}{\geq}\left(\frac{a}{\mathrm{b}}\right)^{\mathrm{b}}-1$$

$$\Leftrightarrow\left(\left(\frac{a}{\mathrm{b}}\right)^{\mathrm{b}}-1\right)\left(\mathrm{b}^{a-\mathrm{b}}-1\right)\boxed{\underset{(**)}{\overset{?}{\geq}}}0$$

$$\text{For } a\geq \mathrm{b}>1,\text{ we have : } \mathrm{b}\left(\ln\frac{a}{\mathrm{b}}\right)\geq 0\text{ and }(a-\mathrm{b})(\ln \mathrm{b})\geq 0$$

$$\Rightarrow\left(\left(\frac{a}{\mathrm{b}}\right)^{\mathrm{b}}-1\right),\left(\mathrm{b}^{a-\mathrm{b}}-1\right)\geq 0\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\text{For } \mathrm{b}>1>a>0,\text{ we have : } \mathrm{b}\left(\ln\frac{a}{\mathrm{b}}\right)\leq 0\text{ and }(a-\mathrm{b})(\ln \mathrm{b})\leq 0$$

$$\Rightarrow\left(\left(\frac{a}{\mathrm{b}}\right)^{\mathrm{b}}-1\right),\left(\mathrm{b}^{a-\mathrm{b}}-1\right)\leq 0\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$\boxed{\text{Case 2}}$ $a>1>b>0$ or $\mathrm{b}\geq a>1$ and then : $a^{a}+\mathrm{b}^{\mathrm{b}}\overset{?}{\geq}a^{\mathrm{b}}+\mathrm{b}^{a}$

$$\Leftrightarrow a^{a-\mathrm{b}}.\left(1-\left(\frac{\mathrm{b}}{a}\right)^{a}\right)\boxed{\underset{(\bullet)}{\overset{?}{\geq}}}1-\left(\frac{\mathrm{b}}{a}\right)^{\mathrm{b}};\text{Now, }(\mathrm{b}-a)\text{ and }(\ln \mathrm{b}-\ln a)\text{ have same sign}$$

$$\therefore(\mathrm{b}-a)\left(\ln\frac{\mathrm{b}}{a}\right)\geq 0\Rightarrow \mathrm{b}\left(\ln\frac{\mathrm{b}}{a}\right)\geq a\left(\ln\frac{\mathrm{b}}{a}\right)\Rightarrow 1-\left(\frac{\mathrm{b}}{a}\right)^{a}\geq 1-\left(\frac{\mathrm{b}}{a}\right)^{\mathrm{b}}\Rightarrow$$

$$a^{a-\mathrm{b}}\left(1-\left(\frac{\mathrm{b}}{a}\right)^{a}\right)\geq a^{a-\mathrm{b}}.\left(1-\left(\frac{\mathrm{b}}{a}\right)^{\mathrm{b}}\right)\overset{?}{\geq}1-\left(\frac{\mathrm{b}}{a}\right)^{\mathrm{b}}\Leftrightarrow\left(1-\left(\frac{\mathrm{b}}{a}\right)^{\mathrm{b}}\right)\left(a^{a-\mathrm{b}}-1\right)\boxed{\underset{(\bullet\bullet)}{\overset{?}{\geq}}}0$$

$$\text{For } a>1>b>0,\text{ we have : } 0\geq \mathrm{b}\left(\ln\frac{\mathrm{b}}{a}\right)\text{ and }(a-\mathrm{b})(\ln a)\geq 0$$

$$\Rightarrow\left(1-\left(\frac{\mathrm{b}}{a}\right)^{\mathrm{b}}\right),\left(a^{a-\mathrm{b}}-1\right)\geq 0\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)\text{ is true}$$

$$\text{For } \mathrm{b}\geq a>1,\text{ we have : } \mathrm{b}\left(\ln\frac{\mathrm{b}}{a}\right)\geq 0\text{ and }(a-\mathrm{b})(\ln a)\leq 0$$

$$\Rightarrow\left(1-\left(\frac{\mathrm{b}}{a}\right)^{\mathrm{b}}\right),\left(a^{a-\mathrm{b}}-1\right)\leq 0\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)\text{ is true}$$

$\boxed{\text{Case 3}}$ $1>a\geq \mathrm{b}>0$ and then : $a^{a}+\mathrm{b}^{\mathrm{b}}\overset{?}{\geq}a^{\mathrm{b}}+\mathrm{b}^{a}\Leftrightarrow a^{a}-a^{\mathrm{b}}\overset{?}{\geq}\mathrm{b}^{a}-\mathrm{b}^{\mathrm{b}}$

Let $\mathrm{f}(x)=x^{a}-x^{\mathrm{b}}$ and so, we require to prove : $\mathrm{f}(a)\overset{?}{\geq}\mathrm{f}(\mathrm{b})$ and we have $\mathrm{f}'(x)=ax^{a-1}-\mathrm{b}x^{\mathrm{b}-1}=x^{\mathrm{b}-1}\left(ax^{a-\mathrm{b}}-\mathrm{b}\right)\rightarrow(2)$

Now, $(\mathrm{b}-1)\ln x>0$ $(\because 1>a\geq x\geq \mathrm{b}>0\Rightarrow(\mathrm{b}-1),\ln x<0)\Rightarrow x^{\mathrm{b}-1}>1>0$ and hence (2) $\Rightarrow$ in order to prove : $\mathrm{f}'(x)\geq 0$, it suffices to prove :

$$ax^{a-\mathrm{b}}-\mathrm{b}\overset{?}{\geq}0\text{ and }(a-\mathrm{b})(\ln x-\ln \mathrm{b})\geq 0\therefore x^{a-\mathrm{b}}\geq \mathrm{b}^{a-\mathrm{b}}\text{ and }\because a>0$$

$$\therefore \text{ it suffices to prove : } ab^{a-b} \overset{?}{\geq} b \Leftrightarrow a \boxed{\underset{(i)}{\overset{?}{\geq}}} b^{1-a+b}$$

$$\text{Let } a = b^m \text{ and then : } m - 1 = \frac{\ln a}{\ln b} - 1 = \frac{\ln a - \ln b}{\ln b} \leq 0 \; (\because a \geq b \text{ and } b < 1)$$

$$\Rightarrow m \leq 1 \rightarrow (3) \text{ and now, (i)} \Leftrightarrow b^m \overset{?}{\geq} b^{1-b^m+b} \Leftrightarrow m \ln b \overset{?}{\geq} (1 - b^m + b)\ln b$$

$$\Leftrightarrow m \overset{?}{\leq} 1 - b^m + b \; (\because b < 1 \Rightarrow \ln b < 0) \Leftrightarrow b^m \boxed{\underset{(ii)}{\overset{?}{\leq}}} 1 + b - m$$

Indeed via (3) and Bernoulli, $b^m \leq 1 + (b-1)m \overset{m \leq 1}{\leq} 1 - m + b \Rightarrow \text{(ii)} \Rightarrow \text{(i)}$ is true $\Rightarrow f'(x) \geq 0 \Rightarrow f(x) \uparrow$ on $[b, a] \Rightarrow f(a) \geq f(b) \Rightarrow a^a + b^b \geq a^b + b^a$

$\boxed{\text{Case 4}}$ $1 > b \geq a > 0$ and then : Let $F(x) = x^b - x^a$ & so, we require to prove :

$$F(b) \overset{?}{\geq} F(a) \text{ and we have } F'(x) = bx^{b-1} - ax^{a-1} = x^{a-1}\left(bx^{b-a} - a\right) \rightarrow (4)$$

$$\text{Now, } x^{a-1} > 0 \therefore (4) \Rightarrow \text{to prove : } F'(x) \geq 0, \text{ suffices to prove : } bx^{b-a} \overset{?}{\geq} a$$

$$\text{and } (b-a)(\ln x - \ln a) \geq 0 \therefore x^{b-a} \geq a^{b-a} \therefore \text{ it suffices to prove : } ba^{b-a} \overset{?}{\geq} a$$

$$\Leftrightarrow b \boxed{\underset{(m)}{\overset{?}{\geq}}} a^{1-b+a}; \text{ Let } b = a^n; \text{ then : } n - 1 = \frac{\ln b - \ln a}{\ln a} \leq 0 \rightarrow (5) \text{ and now, (m)} \Leftrightarrow$$

$$a^n \overset{?}{\geq} a^{1-a^n+a} \Leftrightarrow n \ln a \overset{?}{\geq} (1 - a^n + a)\ln a \Leftrightarrow n \overset{?}{\leq} 1 - a^n + a \Leftrightarrow a^n \boxed{\underset{(n)}{\overset{?}{\leq}}} 1 + a - n$$

Bernoulli, (5) $\Rightarrow a^n \leq 1 + (a-1)n \overset{n \leq 1}{\leq} 1 - n + a \Rightarrow$ (n) is true $\Rightarrow F(x) \uparrow$ on $[a, b]$ $\Rightarrow F(b) \geq F(a) \therefore$ with all cases, $a^a + b^b \geq a^b + b^a$ & analogs $\therefore 2\left(a^a + b^b + c^c\right)$ $\geq a^b + a^c + b^a + b^c + c^a + c^b \; \forall \, a, b, c > 0$, " = " iff $a = b = c$ (QED)