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JP.602 If $a, b, c > 0$ and $\lambda \geq 0$ then:

$$2\sum\frac{a^3}{b+\lambda c}+\frac{\lambda+1}{2}\sum bc \geq 2\sum a^2$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

Using means inequality, we have:

$$\frac{2a^3}{b+\lambda c}+\frac{a(b+\lambda c)}{2} \geq 2\sqrt{\frac{2a^3}{b+\lambda c}\cdot\frac{a(b+\lambda c)}{2}} = 2a^2,$$

with equality for $\frac{2a^3}{b+\lambda c}=\frac{a(b+\lambda c)}{2} \Leftrightarrow 2a = b+\lambda c.$

Writing the other analogs inequalities and summing we obtain:

$$2\sum\frac{a^3}{b+\lambda c}+\sum\frac{a(b+\lambda c)}{2} \geq 2\sum a^2 \Leftrightarrow 2\sum\frac{a^3}{b+\lambda c}+\frac{\lambda+1}{2}\sum bc \geq 2\sum a^2$$

Equality holds if and only if $a=b=c$ and $\lambda = 1$.

Solution 2 by Omkumar Rao-India

$$\sum_{cyc}\frac{a^3}{b+\lambda c}=\sum_{cyc}\frac{a^4}{ab+\lambda ac}$$

$$\textit{Applying Bergstrom's inequality}:$$

$$\sum_{cyc}\frac{a^4}{ab+\lambda ac} \geq \frac{(\sum a^2)^2}{\lambda\sum ab+\sum ab}$$

$$2\sum_{cyc}\frac{a^3}{b+\lambda c} \geq 2\frac{(\sum a^2)^2}{(\lambda+1)\sum ab}$$

$$2\sum_{cyc}\frac{a^3}{b+\lambda c}+\frac{\lambda+1}{2}\sum ab \geq 2\frac{(\sum a^2)^2}{(\lambda+1)\sum ab}+\frac{\lambda+1}{2}\sum ab$$

$$\textit{Applying AM-GM Inequality on RHS}:$$

$$2\frac{(\sum a^2)^2}{(\lambda+1)\sum ab}+\frac{\lambda+1}{2}\sum ab \geq 2\sqrt{\frac{2(\sum a^2)^2}{(\lambda+1)\sum ab}\,\frac{\lambda+1}{2}\sum ab}$$

$$2\sum_{cyc}\frac{a^3}{b+\lambda c}+\frac{\lambda+1}{2}\sum ab \geq 2\sum a^2$$

$$\textit{Therefore}, LHS \geq 2\sum a^2 = RHS$$