ROMANIAN MATHEMATICAL MAGAZINE

JP.601 If $a, b, c > 0$ and $\lambda \geq 0$ then:

$$(\lambda+1)\sum \frac{a^2}{b+\lambda c}+2\sqrt{3(ab+bc+ca)} \geq 3(a+b+c)$$

Proposed by Marin Chirciu – Romania

Solution 1 by proposer

We use pqr Method. We denote $p = a+b+c; q = ab+bc+ca, r = abc$.

Due to homogeneity we can assume $ab+bc+ca = 3$.

We obtain

$$\sum \frac{a^2}{b+\lambda c}=\sum \frac{a^3}{a(b+\lambda c)} \overset{Holder}{\geq} \frac{(\sum a)^3}{3\sum a(b+\lambda c)}=\frac{p^3}{3\cdot(\lambda+1)\sum bc}=\frac{p^3}{3(\lambda+1)q}=$$

$$=\frac{p^3}{3(\lambda+1)\cdot 3}=\frac{p^3}{9(\lambda+1)}.$$

It suffices to prove that:

$$(\lambda+1)\cdot\frac{p^3}{9(\lambda+1)}+2\sqrt{3}\cdot 3 \geq 3p \Leftrightarrow \frac{p^3}{9}+6 \geq 3p \Leftrightarrow p^3-27p+54 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (p-3)^2(p+6) \geq 0.$$

Equality holds if and only if $a = b = c$.

Note: For $\lambda = 1$ we obtain the propose problem by Nguyen Viet Hung in Pure Inequalities 3/2022: If $a, b, c > 0$ then

$$\sum \frac{a^2}{b+c}+\sqrt{3(ab+bc+ca)} \geq \frac{3}{2}(a+b+c)$$

Nguyen Viet Hung – Vietnam

Solution 2 by Soumava Chakraborty=Kolkata-India

$$(\lambda+1)\sum_{cyc}\frac{a^2}{b+\lambda c}+2\sqrt{3\sum_{cyc}ab}=(\lambda+1)\sum_{cyc}\frac{a^3}{ab+\lambda ca}+2\sqrt{3\sum_{cyc}ab}$$

$$\overset{Holder}{\geq}(\lambda+1).\frac{\left(\sum_{cyc}a\right)^3}{3(\lambda+1)\sum_{cyc}ab}+2\sqrt{3\sum_{cyc}ab}=\frac{\left(\sum_{cyc}a\right)^3}{3\sum_{cyc}ab}+\sqrt{3\sum_{cyc}ab}+\sqrt{3\sum_{cyc}ab}$$

$$\overset{\text{AM-GM}}{\geq} 3.\sqrt[3]{\frac{\left(\sum_{\text{cyc}} a\right)^3}{3\sum_{\text{cyc}} ab}\sqrt{3\sum_{\text{cyc}} ab}\,.\sqrt{3\sum_{\text{cyc}} ab}} = 3\sum_{\text{cyc}} a$$

$$\therefore (\lambda+1)\sum_{\text{cyc}}\frac{a^2}{b+\lambda c} + 2\sqrt{3(ab+bc+ca)} \geq 3(a+b+c)\ \forall\, a,b,c > 0 \text{ and } \lambda \geq 0,$$

$$'' ='' \text{ iff } a = b = c \text{ (QED)}$$