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J.3391. *Proposed by Marin Chirciu, Romania.* In any triangle ABC with incenter I and bisectors AD, BE, CF, for all $n \in \mathbb{N}$ it holds:

$$\left(\frac{AI}{AD}\right)^n + \left(\frac{BI}{BD}\right)^n + \left(\frac{CI}{CD}\right)^n \geq 3 \cdot \left(\frac{2}{3}\right)^n.$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. I think there is a typo in the LHS of the stated inequality. I suggest and solved the following corrected statement.

J.3391. (corrected) *Proposed by Marin Chirciu, Romania.* In any triangle ABC with incenter I and internal angle bisectors AD, BE, CF, prove that for all $n \in \mathbb{N}$ it holds:

$$\left(\frac{AI}{AD}\right)^n + \left(\frac{BI}{BE}\right)^n + \left(\frac{CI}{CF}\right)^n \geq 3 \cdot \left(\frac{2}{3}\right)^n$$

Let the side lengths of $\triangle ABC$ opposite to vertices A, B, C be a, b, c respectively. By the Angle Bisector Theorem applied to $\triangle ABC$, the segment AD divides the side BC, and we have

$$\frac{BD}{DC} = \frac{c}{b} \Rightarrow BD = \frac{c}{b+c} \cdot a = \frac{ac}{b+c}$$

Next, considering $\triangle ABD$ where BI acts as the internal angle bisector of $\angle B$, we apply the Angle Bisector Theorem again, to obtain

$$\frac{AI}{ID} = \frac{AB}{BD} = \frac{c}{\frac{ac}{b+c}} = \frac{b+c}{a}$$

Since the total length of the bisector is $AD = AI + ID$, we can find the ratio of the segment to the whole line. That is,

$$\frac{AI}{AD} = \frac{AI}{AI + ID} = \frac{\frac{b+c}{a}}{\frac{b+c}{a} + 1} = \frac{b+c}{a+b+c}$$

By symmetric reasoning for the other two angle bisectors, BE and CF, we obtain

$$\frac{BI}{BE} = \frac{a+c}{a+b+c} \quad \text{and} \quad \frac{CI}{CF} = \frac{a+b}{a+b+c}$$

Adding up these three fundamental ratios together yields

$$\frac{AI}{AD} + \frac{BI}{BE} + \frac{CI}{CF} = \frac{(b+c)+(a+c)+(a+b)}{a+b+c} = \frac{2(a+b+c)}{a+b+c} = 2$$

For any positive real numbers x, y, z and any natural number $n \in \mathbb{N}$, Jensen's Inequality for the convex function $f(t) = t^n$ states that:

$$\frac{x^n + y^n + z^n}{3} \geq \left(\frac{x+y+z}{3}\right)^n$$

By substituting $x = \dfrac{AI}{AD}$, $y = \dfrac{BI}{BE}$, and $z = \dfrac{CI}{CF}$, and using our previous result $x + y + z = 2$, we get

$$\frac{\left(\frac{AI}{AD}\right)^n + \left(\frac{BI}{BE}\right)^n + \left(\frac{CI}{CF}\right)^n}{3} \geq \left(\frac{2}{3}\right)^n$$

from which it follows

$$\left(\frac{AI}{AD}\right)^n + \left(\frac{BI}{BE}\right)^n + \left(\frac{CI}{CF}\right)^n \geq 3 \cdot \left(\frac{2}{3}\right)^n$$

Equality holds if and only if $a = b = c$, meaning $\triangle ABC$ is equilateral, and the proof is complete.