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J.3389. *Proposed by Marin Chirciu, Romania.* Let $\lambda > 0$ be a real number. Find all real solutions of the equation

$$\lambda x(\lambda x + 1) + \lambda y(\lambda y + 1) = \lambda^2 xy - 1.$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. First, we expand the terms on the left-hand side of the equation, to obtain

$$\lambda^2 x^2 - \lambda^2 xy + \lambda^2 y^2 + \lambda x + \lambda y + 1 = 0$$

Grouping the terms by powers of λ, we can view this as a quadratic equation in terms of the parameter λ:

$$\lambda^2(x^2 - xy + y^2) + \lambda(x + y) + 1 = 0$$

For this equation to yield real values for λ (given that x and y are real numbers), the discriminant Δ of this quadratic expression with respect to λ must be non-negative ($\Delta \geq 0$). Identifying the coefficients: $A = x^2 - xy + y^2$, $B = x + y$ and $C = 1$ the discriminant is given by

$$\Delta = B^2 - 4AC = (x + y)^2 - 4(x^2 - xy + y^2)$$

Expanding and simplifying the expression, we have

$$\Delta = (x^2 + 2xy + y^2) - (4x^2 - 4xy + 4y^2) = -3(x - y)^2$$

We require that $\Delta \geq 0$, which gives us the condition:

$$-3(x - y)^2 \geq 0$$

Since $(x - y)^2 \geq 0$ for all real numbers x and y, multiplying it by -3 means $-3(x - y)^2 \leq 0$. The only way for the inequality to hold true is if the expression equals exactly zero:

$$-3(x - y)^2 = 0 \Rightarrow x - y = 0 \Rightarrow x = y$$

Substituting $y = x$ back into our rearranged quadratic equation, yields

$$\begin{aligned} \lambda^2(x^2 - x^2 + x^2) + \lambda(x + x) + 1 &= 0, \\ \lambda^2 x^2 + 2\lambda x + 1 &= 0, \\ (\lambda x + 1)^2 &= 0. \end{aligned}$$

Taking the square root of both sides:

$$\lambda x + 1 = 0 \Rightarrow \lambda x = -1 \Rightarrow x = -\frac{1}{\lambda}$$

Since $y = x$, it follows that $y = -\frac{1}{\lambda}$. Thus, for any given real number $\lambda > 0$, there exists exactly one unique real solution pair (x, y), which is:

$$\left(-\frac{1}{\lambda}, -\frac{1}{\lambda}\right)$$