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J.3388. *Proposed by Marin Chirciu, Romania.* Let x, y be positive real numbers. Prove that $(x^3 + y^3)^4 \leq 2(x^4 + y^4)^3$.

Solution 1 by José Luis Díaz-Barrero, Barcelona, Spain. We will prove this inequality using the Power Mean Inequality. For any positive real numbers x and y , the power mean of order p is defined as:

$$M_p(x, y) = \left(\frac{x^p + y^p}{2} \right)^{\frac{1}{p}}$$

The Power Mean Inequality states that if $p \leq q$, then $M_p(x, y) \leq M_q(x, y)$. Since $3 \leq 4$, we can directly compare the cubic mean and the quartic mean:

$$\left(\frac{x^3 + y^3}{2} \right)^{\frac{1}{3}} \leq \left(\frac{x^4 + y^4}{2} \right)^{\frac{1}{4}}$$

To eliminate the fractional exponents, we raise both sides of the inequality to the power of 12, and we get

$$\left[\left(\frac{x^3 + y^3}{2} \right)^{\frac{1}{3}} \right]^{12} \leq \left[\left(\frac{x^4 + y^4}{2} \right)^{\frac{1}{4}} \right]^{12}$$

Simplifying the exponents yields:

$$\left(\frac{x^3 + y^3}{2} \right)^4 \leq \left(\frac{x^4 + y^4}{2} \right)^3 \Leftrightarrow (x^3 + y^3)^4 \leq 2(x^4 + y^4)^3,$$

and the inequality holds for all positive real numbers x and y . Equality occurs if and only if $x = y$.

Solution 2 by José Luis Díaz-Barrero, Barcelona, Spain. Since the inequality is homogeneous and both sides have a total degree of 12, we can prove it by expanding both sides algebraically and analyzing the difference RHS–LHS. Using the Binomial Theorem, we expand the LHS:

$$(x^3 + y^3)^4 = x^{12} + 4x^9y^3 + 6x^6y^6 + 4x^3y^9 + y^{12}$$

Using symmetric sum notation \sum_{sym} , this can be written as:

$$(x^3 + y^3)^4 = \frac{1}{2} \sum_{\text{sym}} x^{12} + 4 \sum_{\text{sym}} x^9y^3 + 3 \sum_{\text{sym}} x^6y^6$$

Next, we expand the RHS:

$$2(x^4 + y^4)^3 = 2(x^{12} + 3x^8y^4 + 3x^4y^8 + y^{12}) = 2x^{12} + 6x^8y^4 + 6x^4y^8 + 2y^{12}$$

Using symmetric sum notation, this is:

$$2(x^4 + y^4)^3 = \sum_{\text{sym}} x^{12} + 6 \sum_{\text{sym}} x^8 y^4$$

We subtract the LHS from the RHS to see if the resulting polynomial is non-negative:

$$\text{RHS} - \text{LHS} = \frac{1}{2} \sum_{\text{sym}} x^{12} + 6 \sum_{\text{sym}} x^8 y^4 - 4 \sum_{\text{sym}} x^9 y^3 - 3 \sum_{\text{sym}} x^6 y^6$$

Combining the symmetric sums back into standard polynomial form yields:

$$\text{RHS} - \text{LHS} = x^{12} - 4x^9 y^3 + 6x^8 y^4 - 6x^6 y^6 + 6x^4 y^8 - 4x^3 y^9 + y^{12}$$

By grouping and factoring, this expression becomes

$$\text{RHS} - \text{LHS} = (x - y)^4 (x + y)^2 (2x^6 + 4x^5 y + 7x^4 y^2 + 8x^3 y^3 + 7x^2 y^4 + 4x y^5 + 2y^6)$$

Since x and y are positive real numbers:

1. $(x - y)^4 \geq 0$ (as any real number raised to an even power is non-negative).
2. $(x + y)^2 > 0$.
3. The large polynomial factor consists entirely of positive coefficients and positive terms, so it is strictly greater than 0.

Therefore, $\text{RHS} - \text{LHS} \geq 0$, which means

$$(x^3 + y^3)^4 \leq 2(x^4 + y^4)^3$$

Equality holds if and only if $x = y$.