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J.3383. *Proposed by Marin Chirciu, Romania.* In any triangle ABC, with the usual notations, it holds:

$$3\sum_{cyc} a^2 \tan\frac{A}{2} \geq \sum_{cyc} a^2 \cot\frac{A}{2}.$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. Let $x, y, z > 0$ be the standard substitution variables representing the lengths of the tangents from the vertices to the incircle:

$$a = y+z, \quad b = z+x, \quad c = x+y$$

Let $s = x+y+z$ be the semiperimeter, r be the inradius, and Δ be the area of the triangle. Using standard half-angle identities, we have:

$$\tan\frac{A}{2} = \frac{r}{x}, \quad \tan\frac{B}{2} = \frac{r}{y}, \quad \tan\frac{C}{2} = \frac{r}{z}$$

$$\cot\frac{A}{2} = \frac{x}{r}, \quad \cot\frac{B}{2} = \frac{y}{r}, \quad \cot\frac{C}{2} = \frac{z}{r}$$

Substituting these identities into the original inequality allows us to express the Left-Hand Side (LHS) and Right-Hand Side (RHS) as follows:

$$\text{LHS} = 3r\left[\frac{(y+z)^2}{x} + \frac{(z+x)^2}{y} + \frac{(x+y)^2}{z}\right]$$

$$\text{RHS} = \frac{1}{r}\left[x(y+z)^2 + y(z+x)^2 + z(x+y)^2\right]$$

From Heron's Formula, the inradius satisfies $r^2 = \frac{xyz}{x+y+z}$, which gives $\frac{1}{r} = \frac{r(x+y+z)}{xyz}$. Substituting this into the RHS and dividing both sides of the inequality by r, the problem reduces to proving:

$$3\left[\frac{(y+z)^2}{x} + \frac{(z+x)^2}{y} + \frac{(x+y)^2}{z}\right] \geq \frac{x+y+z}{xyz}\left[x(y+z)^2 + y(z+x)^2 + z(x+y)^2\right]$$

To compare the two expressions accurately, we expand both sides fully using symmetric sums ($\sum_{sym}$) and cyclic sums ($\sum_{cyc}$). Expanding the LHS yields

$$\text{LHS} = 3\left(\frac{y^2+2yz+z^2}{x} + \frac{z^2+2zx+x^2}{y} + \frac{x^2+2xy+y^2}{z}\right)$$

Grouping these individual fractions into algebraic structures, we get

$$\text{LHS} = 3\sum_{sym}\frac{x^2}{y} + 6\sum_{cyc}\frac{xy}{z}$$

Now, we expand the RHS. Distributing the terms within the bracket yields

$$x(y+z)^2 + y(z+x)^2 + z(x+y)^2 = \sum_{sym} x^2y + 6xyz$$

Multiplying this by the leading factor $\frac{x+y+z}{xyz}$ gives:

$$\text{RHS} = (x+y+z)\left[\frac{\sum_{sym} x^2y}{xyz} + 6\right] = (x+y+z)\left[\sum_{sym}\frac{x}{y} + 6\right]$$

Distributing $(x+y+z)$ completely across the symmetric fractional sum results in:

$$\text{RHS} = \sum_{sym}\frac{x^2}{y} + 2\sum_{cyc}\frac{xy}{z} + 8(x+y+z)$$

We subtract the expanded RHS from the expanded LHS to determine the sign of their difference:

$$\text{LHS} - \text{RHS} = \left(3\sum_{sym}\frac{x^2}{y} + 6\sum_{cyc}\frac{xy}{z}\right) - \left(\sum_{sym}\frac{x^2}{y} + 2\sum_{cyc}\frac{xy}{z} + 8(x+y+z)\right)$$

$$\text{LHS} - \text{RHS} = 2\sum_{sym}\frac{x^2}{y} + 4\sum_{cyc}\frac{xy}{z} - 8(x+y+z)$$

To confirm $\text{LHS} - \text{RHS} \geq 0$, we check the validity of the following inequality:

$$\sum_{sym}\frac{x^2}{y} + 2\sum_{cyc}\frac{xy}{z} \geq 4(x+y+z)$$

We break this down into two classical inequalities for positive real numbers:

1. Applying that $\dfrac{x^2}{y} + \dfrac{y^2}{x} \geq x + y \Leftrightarrow (x+y)(x-y)^2 \geq 0$ across all variable pairs yields:

$$\sum_{sym}\frac{x^2}{y} \geq 2(x+y+z)$$

2. Applying the AM-GM inequality cyclically (since $\dfrac{xy}{z} + \dfrac{yz}{x} \geq 2y$, $\dfrac{yz}{x} + \dfrac{zx}{y} \geq 2z$, and $\dfrac{zx}{y} + \dfrac{xy}{z} \geq 2x$) yields:

$$\sum_{cyc}\frac{xy}{z} \geq x+y+z \Rightarrow 2\sum_{cyc}\frac{xy}{z} \geq 2(x+y+z)$$

Summing these two inequalities directly results in:

$$\sum_{sym}\frac{x^2}{y} + 2\sum_{cyc}\frac{xy}{z} \geq 4(x+y+z)$$

Multiplying this entire expression by 2 establishes that $\text{LHS} - \text{RHS} \geq 0$. Thus, the original inequality holds true. Equality holds if and only if $x = y = z$, which corresponds to $a = b = c$ an equilateral triangle.