José Luis **Díaz-Barrero**
Barcelona Mathematical Circle (BMC)
jose.luis.diaz@upc.edu

J.3378. *Proposed by Dumitru Bătineţu-Giurgiu, and Claudia Nănuţi, Romania.* If $a, b, c, x, y > 0$, then prove that

$$\frac{a^2b^2}{c(xa+yb)} + \frac{b^2c^2}{a(xb+yc)} + \frac{c^2a^2}{b(xc+ya)} \geq \frac{ab+bc+ca}{x+y}$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. We use the Cauchy-Schwarz Inequality in the form:

$$\frac{u_1^2}{v_1} + \frac{u_2^2}{v_2} + \frac{u_3^2}{v_3} \geq \frac{(u_1+u_2+u_3)^2}{v_1+v_2+v_3}$$

for positive real numbers u_1, u_2, u_3 and v_1, v_2, v_3. Putting $u_1 = ab$, $u_2 = bc$, $u_3 = ca$, and $v_1 = c(xa+yb)$, $v_2 = a(xb+yc)$, $v_3 = b(xc+ya)$, we get

$$\frac{a^2b^2}{c(xa+yb)} + \frac{b^2c^2}{a(xb+yc)} + \frac{c^2a^2}{b(xc+ya)} \geq \frac{(ab+bc+ca)^2}{c(xa+yb)+a(xb+yc)+b(xc+ya)}$$

Expanding the denominator on the right-hand side, we have

$$\begin{aligned} c(xa+yb)+a(xb+yc)+b(xc+ya) &= xac+ybc+xab+yac+xbc+yab \\ &= x(ab+bc+ca)+y(ab+bc+ca) \\ &= (x+y)(ab+bc+ca) \end{aligned}$$

Substituting the simplified denominator back into our inequality yields

$$\begin{aligned} \text{LHS} &\geq \frac{(ab+bc+ca)^2}{(x+y)(ab+bc+ca)} \\ &= \frac{ab+bc+ca}{x+y}, \end{aligned}$$

and we may conclude that

$$\frac{a^2b^2}{c(xa+yb)} + \frac{b^2c^2}{a(xb+yc)} + \frac{c^2a^2}{b(xc+ya)} \geq \frac{ab+bc+ca}{x+y}$$

Equality holds if and only if $a = b = c$.