José Luis **Díaz-Barrero**
Barcelona Mathematical Circle (BMC)
jose.luis.diaz@upc.edu

J.3365. *Proposed by Necualai Stanciu, Romania.* If $0 < 2a_k \leq 1$ $(k = 1, 2, \ldots, n)$ and $\sum_{k=1}^{n} a_k^2 = 1$, then prove that

$$\sum_{k=1}^{n} \frac{9}{2(1 - a_k)} \leq 5n + 16$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Given $0 < 2a_k \leq 1$, we have $a_k \in \left(0, \frac{1}{2}\right]$ for all $k = 1, 2, \ldots, n$. We claim that for any $a \in \left(0, \frac{1}{2}\right]$, the following inequality holds

$$\frac{9}{2(1 - a)} \leq 16a^2 + 5$$

Rearranging terms, we get

$$2(1 - a)(16a^2 + 5) - 9 \geq 0$$

Expanding the expression, we have

$$\begin{aligned} 2(1 - a)(16a^2 + 5) - 9 &= 2(16a^2 + 5 - 16a^3 - 5a) - 9 \\ &= -32a^3 + 32a^2 - 10a + 10 - 9 \\ &= -32a^3 + 32a^2 - 10a + 1 = (1 - 2a)(4a - 1)^2 \geq 0 \end{aligned}$$

Since $a \leq \frac{1}{2}$, we have $1 - 2a \geq 0$. Since $(4a - 1)^2 \geq 0$ for all real a, their product is non-negative

$$(1 - 2a)(4a - 1)^2 \geq 0$$

Therefore,

$$2(1 - a)(16a^2 + 5) - 9 \geq 0 \Rightarrow \frac{9}{2(1 - a)} \leq 16a^2 + 5$$

Applying this inequality to each term a_k yields

$$\sum_{k=1}^{n} \frac{9}{2(1 - a_k)} \leq \sum_{k=1}^{n} (16a_k^2 + 5)$$

Splitting the sum on the right-hand side yields, we obtain

$$\sum_{k=1}^{n} \frac{9}{2(1 - a_k)} \leq 16 \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} 5$$

We are given that $\sum_{k=1}^{n} a_k^2 = 1$. Substituting this value into the inequality gives

$$\sum_{k=1}^{n} \frac{9}{2(1 - a_k)} \leq 16(1) + 5n = 5n + 16,$$

as desired.

The equality condition comes directly from the factored expression

$$(1-2a_k)(4a_k-1)^2 = 0$$

For equality to hold across the entire sum, every single term in the summation must achieve equality individually. Setting the factored expression to zero for a specific a_k:

$$(1-2a_k)(4a_k-1)^2 = 0$$

This yields two possible roots for each $a_k \in \left(0, \frac{1}{2}\right]$:

- $1-2a_k = 0 \Rightarrow a_k = \frac{1}{2}$
- $4a_k - 1 = 0 \Rightarrow a_k = \frac{1}{4}$

Thus, for each $k \in \{1, 2, \ldots, n\}$, a_k must be either $\frac{1}{4}$ or $\frac{1}{2}$.

Suppose among the n variables, exactly m of them equal $\frac{1}{2}$, and the remaining $n-m$ equal $\frac{1}{4}$ (where $0 \le m \le n$). Substituting these values into the constraint $\sum_{k=1}^{n} a_k^2 = 1$, we have

$$m\left(\frac{1}{2}\right)^2 + (n-m)\left(\frac{1}{4}\right)^2 = 1$$

$$\frac{m}{4} + \frac{n-m}{16} = 1 \Leftrightarrow 3m + n = 16$$

Depending on n, integer solutions (m, n) for $3m + n = 16$ tell us when equality is possible:

- Case 1 $(m = 0, n = 16)$: All $a_k = \frac{1}{4}$ for $n = 16$.

$$\sum_{k=1}^{16}\left(\frac{1}{4}\right)^2 = 16 \cdot \frac{1}{16} = 1$$

- Case 2 $(m = 4, n = 4)$: All $a_k = \frac{1}{2}$ for $n = 4$.

$$\sum_{k=1}^{4}\left(\frac{1}{2}\right)^2 = 4 \cdot \frac{1}{4} = 1$$

- Case 3 $(m = 1, n = 13)$: One variable equals $\frac{1}{2}$ and twelve variables equal $\frac{1}{4}$ for $n = 13$.

$$1 \cdot \left(\frac{1}{2}\right)^2 + 12 \cdot \left(\frac{1}{4}\right)^2 = \frac{1}{4} + \frac{12}{16} = 1$$