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J.3363. *Proposed by Necualai Stanciu, Romania.* If $x, y, z > 0$, then prove that:

$$\sum_{cyc} \frac{(x+y)^5 - x^5 - y^5}{5((x+y)^3 - x^3 - y^3)} \leq \sum_{cyc} x^2$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Expanding $(x+y)^5$ using the binomial theorem yields

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

from which

$$(x+y)^5 - x^5 - y^5 = 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 = 5xy\left(x^3 + 2x^2y + 2xy^2 + y^3\right)$$

Group terms inside the parentheses:

$$\begin{aligned}(x+y)^5 - x^5 - y^5 &= 5xy\left[(x^3+y^3) + 2xy(x+y)\right]\\ &= 5xy\left[(x+y)(x^2 - xy + y^2) + 2xy(x+y)\right]\\ &= 5xy(x+y)(x^2+xy+y^2)\end{aligned}$$

Likewise, expanding $(x+y)^3$, we have

$$(x+y)^3 - x^3 - y^3 = 3x^2y + 3xy^2 = 3xy(x+y) = 5\left((x+y)^3 - x^3 - y^3\right) = 15xy(x+y)$$

Dividing the numerator by the denominator yields

$$\frac{(x+y)^5 - x^5 - y^5}{5((x+y)^3 - x^3 - y^3)} = \frac{5xy(x+y)(x^2+xy+y^2)}{15xy(x+y)} = \frac{x^2+xy+y^2}{3}$$

Adding cyclically over (x, y), (y, z), and (z, x), we obtain

$$\begin{aligned}\sum_{\text{cyc}} \frac{(x+y)^5 - x^5 - y^5}{5((x+y)^3 - x^3 - y^3)} &= \frac{(x^2+xy+y^2) + (y^2+yz+z^2) + (z^2+zx+x^2)}{3}\\ &= \frac{2(x^2+y^2+z^2) + (xy+yz+zx)}{3}\end{aligned}$$

On account that $xy + yz + zx \leq x^2 + y^2 + z^2$, we have

$$\frac{2(x^2+y^2+z^2) + (xy+yz+zx)}{3} \leq \frac{2(x^2+y^2+z^2) + (x^2+y^2+z^2)}{3} = x^2 + y^2 + z^2$$

Thus, we conclude that

$$\sum_{\text{cyc}} \frac{(x+y)^5 - x^5 - y^5}{5((x+y)^3 - x^3 - y^3)} \leq \sum_{\text{cyc}} x^2$$

Equality holds if and only if $x = y = z$.