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J.3362. *Proposed by Mihály Bencze and Neculai Stanciu, Romania.* If $a_k > 0$ ($k = 1, 2, \dots, n$) and $\sum_{k=1}^n \frac{1}{n+1-a_k} = 1$, then prove that

$$\sum_{k=1}^n \frac{1}{n+1-a_k^2} \leq 1.$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. We suggest that the direction of the inequality must be changed. Indeed, to demonstrate that the inequality $\sum_{k=1}^n \frac{1}{n+1-a_k^2} \leq 1$ does not generally hold, we construct a specific counterexample. For $n = 2$, the given constraint simplifies to:

$$\frac{1}{3-a_1} + \frac{1}{3-a_2} = 1$$

Let us choose a valid positive value for a_1 , say $a_1 = 1/2$. Substituting this into the constraint allows us to solve for a_2 :

$$\frac{1}{3-1/2} + \frac{1}{3-a_2} = 1 \Leftrightarrow \frac{1}{5/2} + \frac{1}{3-a_2} = 1 \Leftrightarrow \frac{2}{5} + \frac{1}{3-a_2} = 1 \Rightarrow a_2 = \frac{4}{3}$$

Both chosen values satisfy the initial conditions since $a_1 = 1/2 > 0$ and $a_2 = 4/3 > 0$. Now, substituting $a_1 = 1/2$ and $a_2 = 4/3$ into the squared target expression $\sum_{k=1}^2 \frac{1}{3-a_k^2}$ yields

$$\sum_{k=1}^2 \frac{1}{3-a_k^2} = \frac{1}{3-(1/2)^2} + \frac{1}{3-(4/3)^2} = \frac{13}{11}$$

Since $\frac{13}{11} > 1$, we have explicitly found a case where:

$$\sum_{k=1}^n \frac{1}{n+1-a_k^2} > 1$$

This disproves the claim that the sum is always less than or equal to 1, confirming that the correct direction must be ≥ 1 , as we have changed in the modify statement.

I propose the following modify statement:

If $0 < a_k < \sqrt{n+1}$ for $k = 1, 2, \dots, n$ and

$$\sum_{k=1}^n \frac{1}{n+1-a_k} = 1$$

then prove that:

$$\sum_{k=1}^n \frac{1}{n+1-a_k^2} \geq 1$$

Let us define $x_k = \frac{1}{n+1-a_k}$. Since $0 < a_k < \sqrt{n+1} \leq n+1$, the denominators are strictly positive ($n+1-a_k > 0$), which implies $x_k > 0$ for all k . Furthermore, because these positive terms sum to exactly 1, each individual component must be strictly less than the total sum, yielding $0 < x_k < 1$.

Solving for a_k in terms of x_k gives:

$$n+1-a_k = \frac{1}{x_k} \implies a_k = n+1 - \frac{1}{x_k}$$

Since $a_k > 0$, we have $n+1 - \frac{1}{x_k} > 0$, which implies $x_k > \frac{1}{n+1}$.

To establish a lower bound for the target sum, we introduce a function $f(x)$ defined on the interval $x \in (\frac{1}{n+1}, 1)$, which maps each x_k to its corresponding term in the target sum:

$$f(x) = \frac{1}{n+1 - \left(n+1 - \frac{1}{x}\right)^2}$$

The symmetric equilibrium point occurs when $a_1 = a_2 = \dots = a_n = 1$, which corresponds to $x = 1/n$. At this point, we have:

$$f\left(\frac{1}{n}\right) = \frac{1}{n+1 - (n+1-n)^2} = \frac{1}{n}$$

Next, we find the tangent line to $f(x)$ at $x = 1/n$ by calculating the derivative $f'(x)$:

$$f'(x) = \frac{2\left(n+1 - \frac{1}{x}\right)}{x^2 \left[n+1 - \left(n+1 - \frac{1}{x}\right)^2\right]^2}$$

Evaluating the derivative at $x = 1/n$ yields:

$$f'\left(\frac{1}{n}\right) = \frac{2(1)}{\left(\frac{1}{n}\right)^2 [n+1-1]^2} = \frac{2}{\frac{1}{n^2} \cdot n^2} = 2$$

Thus, the equation of the tangent line at $x = 1/n$ is:

$$y = f'\left(\frac{1}{n}\right) \left(x - \frac{1}{n}\right) + f\left(\frac{1}{n}\right) = 2 \left(x - \frac{1}{n}\right) + \frac{1}{n} = 2x - \frac{1}{n}$$

To prove that $f(x) \geq 2x - \frac{1}{n}$, we analyze their difference. Let $u = n+1 - \frac{1}{x}$ (where u represents any a_k , meaning $0 < u < \sqrt{n+1}$). Then:

$$\begin{aligned} f(x) - \left(2x - \frac{1}{n}\right) &= \frac{1}{n+1-u^2} - \left(\frac{2}{n+1-u} - \frac{1}{n}\right) \\ &= \frac{1}{n+1-u^2} - \frac{n-1+u}{n(n+1-u)} \\ &= \frac{n(n+1-u) - (n-1+u)(n+1-u^2)}{n(n+1-u)(n+1-u^2)} \end{aligned}$$

Expanding and factoring the numerator, we get:

$$n(n+1-u) - (n-1+u)(n+1-u^2) = (u-1)^2(u+n+1)$$

Therefore, we obtain

$$f(x) - \left(2x - \frac{1}{n}\right) = \frac{(u-1)^2(u+n+1)}{n(n+1-u)(n+1-u^2)}$$

Now, let us examine the signs of the terms within the domain $0 < u < \sqrt{n+1}$:

1. $(u-1)^2 \geq 0$ as it is a perfect square.
2. $u+n+1 > 0$ because $u > 0$ and $n \geq 1$.
3. $n(n+1-u) > 0$ because $u < \sqrt{n+1} \leq n+1$.
4. $n+1-u^2 > 0$ because $u < \sqrt{n+1} \implies u^2 < n+1$.

Since all individual components of the expression are strictly positive, it follows that:

$$f(x) \geq 2x - \frac{1}{n}$$

Substituting $x = x_k$ and summing this inequality for all $k = 1, 2, \dots, n$:

$$\sum_{k=1}^n \frac{1}{n+1-a_k^2} \geq \sum_{k=1}^n \left(2x_k - \frac{1}{n}\right) = 2 \sum_{k=1}^n x_k - n \left(\frac{1}{n}\right)$$

Using the given hypothesis that $\sum_{k=1}^n x_k = 1$, we conclude:

$$\sum_{k=1}^n \frac{1}{n+1-a_k^2} \geq 2(1) - 1 = 1$$

Equality holds if and only if $u = 1$ for all terms, which means $a_1 = a_2 = \dots = a_n = 1$.