José Luis **Díaz-Barrero**
Barcelona Mathematical Circle (BMC)
jose.luis.diaz@upc.edu

J.3361. *Proposed by Neculai Stanciu, Romania.* If $a_k > 0$ $(k = 1, 2, \ldots, n)$, then prove that

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} a_k^{n-1}\right) - \sum_{k=1}^{n} a_k^n - n(n-1)\prod_{k=1}^{n} a_k \geq 0.$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. We begin by expanding the first product in the inequality. Distributing the terms yields two distinct groups: terms where the indices coincide $(i = j)$ and terms where the indices are distinct $(i \neq j)$. Thus, we have:

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} a_k^{n-1}\right) = \sum_{k=1}^{n} a_k^n + \sum_{i \neq j} a_i a_j^{n-1}$$

Substituting this expansion back into the original inequality allows us to cancel the $\sum_{k=1}^{n} a_k^n$ terms. The inequality is therefore equivalent to:

$$\sum_{i \neq j} a_i a_j^{n-1} - n(n-1)\prod_{k=1}^{n} a_k \geq 0$$

or, equivalently:

$$\sum_{i \neq j} a_i a_j^{n-1} \geq n(n-1)\prod_{k=1}^{n} a_k$$

The summation on the left-hand side contains exactly $n(n-1)$ positive terms. To apply the Arithmetic Mean-Geometric Mean (AM-GM) inequality, we compute the product of all these terms. By symmetry, each variable a_m (for $1 \leq m \leq n$) appears in this product with a total exponent determined as follows:

- a_m appears as a_i (with exponent 1) a total of $n-1$ times (whenever $j \neq m$).
- a_m appears as a_j^{n-1} (with exponent $n-1$) a total of $n-1$ times (whenever $i \neq m$).

Thus, the total exponent for each a_m is:

$$(n-1) \cdot 1 + (n-1)(n-1) = (n-1)(1 + n - 1) = n(n-1)$$

It follows that the product of all $n(n-1)$ terms is:

$$\prod_{i \neq j} a_i a_j^{n-1} = \prod_{m=1}^{n} a_m^{n(n-1)} = \left(\prod_{k=1}^{n} a_k\right)^{n(n-1)}$$

By the AM-GM inequality, the arithmetic mean of these $n(n-1)$ terms is greater than or equal to their geometric mean:

$$\frac{1}{n(n-1)} \sum_{i \neq j} a_i a_j^{n-1} \geq \sqrt[n(n-1)]{\left(\prod_{k=1}^{n} a_k \right)^{n(n-1)}}$$

Simplifying the right-hand side yields:

$$\frac{1}{n(n-1)} \sum_{i \neq j} a_i a_j^{n-1} \geq \prod_{k=1}^{n} a_k$$

Multiplying both sides by $n(n-1)$, we obtain:

$$\sum_{i \neq j} a_i a_j^{n-1} \geq n(n-1) \prod_{k=1}^{n} a_k$$

This establishes the required inequality. By the properties of the AM-GM inequality, equality holds if and only if all $n(n-1)$ terms are equal, which implies $a_1 = a_2 = \ldots = a_n$.