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J.3360. *Proposed by Mihály Bencze and Necualai Stanciu, Romania.* In all non isosceles triangles ABC, the following holds:

$$\sum_{cyc} \frac{\sin\frac{A}{2}}{\sin^2 A \sin\frac{A-B}{2}\sin\frac{A-C}{2}} = \frac{R(s^2+r^2+4Rr)}{s}$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. I think there is a typo in the statement. If we take $A = 30°, B = 40°, C = 110°$ we have $LHS \simeq 4.43$, $RHS \simeq 2.68$ and they do not match. I suggest the following corrected identity:

In all non isosceles triangles ABC, the following holds:

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Let us analyze the first term of the cyclic sum, denoted as T_A:

$$T_A = \frac{\sin\frac{A}{2}}{\sin^2 A \sin\frac{A-B}{2}\sin\frac{A-C}{2}}$$

Using the double-angle identity for the sine function, $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$, we can expand the denominator's $\sin^2 A$ term as

$$\sin^2 A = 4\sin^2\frac{A}{2}\cos^2\frac{A}{2}$$

Substituting this back into T_A allows us to cancel out one factor of $\sin\frac{A}{2}$ to obtain

$$T_A = \frac{1}{4\sin\frac{A}{2}\cos^2\frac{A}{2}\sin\frac{A-B}{2}\sin\frac{A-C}{2}}$$

By applying product-to-sum trigonometric identities to the terms $\sin\dfrac{A-B}{2}\sin\dfrac{A-C}{2}$ and using the half-angle relations for a triangle (where $\frac{A+B+C}{2} = \frac{\pi}{2}$), we can find a common cyclic denominator

$$D = \sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2}$$

When we sum all three cyclic terms ($T_A+T_B+T_C$) over this common denominator and convert the trigonometric functions into algebraic expressions of the side lengths a, b, c via the Law of Sines ($a = 2R\sin A$), the numerator simplifies to a symmetric polynomial representation.

To explicitly derive the transition from the trigonometric cyclic sum to the algebraic formulation, we utilize Mollweide's Formula and standard half-angle identities.

1. Transforming the Angle Differences via Mollweide's Formula. Mollweide's formula relates the differences of two angles to the sides of a triangle:

$$\sin\frac{A-B}{2} = \frac{a-b}{c}\cos\frac{C}{2}$$

Applying this to the two difference terms in the denominator of T_A, we get:

$$\begin{aligned}\sin\frac{A-B}{2}\sin\frac{A-C}{2} &= \left(\frac{a-b}{c}\cos\frac{C}{2}\right)\left(\frac{a-c}{b}\cos\frac{B}{2}\right)\\ &= \frac{(a-b)(a-c)}{bc}\cos\frac{B}{2}\cos\frac{C}{2}\end{aligned}$$

2. Eliminating the Half-Angles. Using the standard geometric formulas for half-angles in terms of the semiperimeter s:

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}, \quad \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

Multiplying the two cosine terms yields:

$$\cos\frac{B}{2}\cos\frac{C}{2} = \sqrt{\frac{s^2(s-b)(s-c)}{a^2bc}} = \frac{s}{a}\sqrt{\frac{(s-b)(s-c)}{bc}} = \frac{s}{a}\sin\frac{A}{2}$$

Rearranging this gives an identity that simplifies our numerator's half-angle to

$$\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{a}{s}$$

Now, substitute these expressions back into the full definition of T_A, noting that from the Law of Sines, $\sin^2 A = \frac{a^2}{4R^2}$:

$$T_A = \frac{1}{\sin^2 A}\cdot\frac{bc}{(a-b)(a-c)}\cdot\left(\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}\right)$$

$$T_A = \frac{4R^2}{a^2}\cdot\frac{bc}{(a-b)(a-c)}\cdot\frac{a}{s} = \frac{4R^2bc}{s\cdot a(a-b)(a-c)}$$

To create a uniform structural expression across all terms, we multiply the numerator and denominator by a:

$$T_A = \frac{4R^2abc}{s\cdot a^2(a-b)(a-c)} = -\frac{4R^2abc(b-c)}{s\cdot a^2(a-b)(b-c)(c-a)}$$

3. Summing Cyclically Over the Common Denominator. Factoring out the shared global components from the cyclic sum $\sum_{cyc} T_A$, we obtain:

$$\sum_{cyc} T_A = -\frac{4R^2abc}{s(a-b)(b-c)(c-a)}\sum_{cyc}\frac{b-c}{a^2}$$

By finding a common denominator for the inner algebraic cyclic sum $\sum_{cyc}\frac{b-c}{a^2}$, the inner numerator simplifies via polynomial factorization:

$$\sum_{cyc}\frac{b-c}{a^2} = \frac{(b-c)b^2c^2 + (c-a)a^2c^2 + (a-b)a^2b^2}{a^2b^2c^2}$$

$$= \frac{-(a-b)(b-c)(c-a)(ab+bc+ca)}{(abc)^2}$$

Substituting this identity back into our main equation leads to a dramatic cancellation of the non-isosceles difference boundaries:

$$\sum_{cyc} T_A = -\frac{4R^2abc}{s(a-b)(b-c)(c-a)} \cdot \left[\frac{-(a-b)(b-c)(c-a)(ab+bc+ca)}{(abc)^2}\right]$$

$$\sum_{cyc} T_A = \frac{4R^2(ab+bc+ca)}{s \cdot abc}$$

Using the geometric identity for the product of triangle side lengths, $abc = 4R\Delta = 4Rrs$, we can clear the final variables to obtain

$$\sum_{cyc} T_A = \frac{4R^2(ab+bc+ca)}{s \cdot (4Rrs)} = \frac{R(ab+bc+ca)}{s^2r}$$

Since $ab+bc+ca = s^2+r^2+4Rr$, we have

$$\sum_{cyc} \frac{\sin\frac{A}{2}}{\sin^2 A \sin\frac{A-B}{2} \sin\frac{A-C}{2}} = \frac{R(s^2+r^2+4Rr)}{s^2r},$$

as claimed.