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J.3359. *Proposed by Mihály Bencze and Necualai Stanciu, Romania.* If $a_k > 0$ for $k = 1, 2, \ldots, n$ and $\sum_{k=1}^{n} a_k^2 = n$, then prove that

$$\sum_{k=1}^{n} \frac{1}{n+1-a_k} \leq 1$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. We can solve this problem using the Tangent Line Method by establishing a quadratic upper bound for the terms when $n \geq 3$, and handling the small base cases separately.

Case 1: $n = 1$

For $n = 1$, the constraint becomes $a_1^2 = 1$. Since $a_1 > 0$, we have $a_1 = 1$. Substituting this into the inequality:

$$\frac{1}{1+1-1} = 1 \leq 1$$

Thus, the inequality holds as an equality.

Case 2: $n = 2$

For $n = 2$, the constraint is $a_1^2 + a_2^2 = 2$. We wish to prove:

$$\frac{1}{3-a_1} + \frac{1}{3-a_2} \leq 1$$

Combining the fractions on the left-hand side yields:

$$\frac{(3-a_2)+(3-a_1)}{(3-a_1)(3-a_2)} \leq 1 \implies \frac{6-(a_1+a_2)}{9-3(a_1+a_2)+a_1a_2} \leq 1$$

Since $a_1, a_2 \leq \sqrt{2} < 3$, the denominator is strictly positive. Cross-multiplying gives:

$$6-(a_1+a_2) \leq 9-3(a_1+a_2)+a_1a_2$$
$$2(a_1+a_2)-a_1a_2 \leq 3$$

Let $s = a_1 + a_2$ and $p = a_1a_2$. Using the identity $s^2 - 2p = a_1^2 + a_2^2 = 2$, we find $p = \frac{s^2-2}{2}$. Substituting this into the inequality:

$$2s - \frac{s^2-2}{2} \leq 3 \implies 4s - s^2 + 2 \leq 6 \implies s^2 - 4s + 4 \geq 0 \implies (s-2)^2 \geq 0$$

Since a perfect square is always non-negative, the case for $n = 2$ is proven.

Case 3: $n \geq 3$

For the general case where $n \geq 3$, we establish a quadratic upper bound for each individual term using the Tangent Line Method:

$$\frac{1}{n+1-x} \leq \frac{x^2+2n-1}{2n^2}$$

We carry out the derivation of the Quadratic Upper Bound. To understand where the expression $\dfrac{x^2+2n-1}{2n^2}$ comes from, we use the tangent line method (adapted for a quadratic curve). When working with a symmetric constraint like $\sum a_k^2 = n$, the equality case occurs when all variables are equal, meaning $a_1 = a_2 = \cdots = a_n = 1$. Therefore, we evaluate our target function $f(x) = \dfrac{1}{n+1-x}$ and its derivative at the center point $x = 1$.

First, we find the value of the function at $x = 1$ to obtain

$$f(1) = \frac{1}{n+1-1} = \frac{1}{n}$$

Next, we calculate the first derivative of $f(x)$:

$$f'(x) = \frac{d}{dx}\left((n+1-x)^{-1}\right) = -(1)(n+1-x)^{-2} \cdot (-1) = \frac{1}{(n+1-x)^2}$$

Evaluating this derivative at $x = 1$ gives

$$f'(1) = \frac{1}{(n+1-1)^2} = \frac{1}{n^2}$$

Because our constraint involves x^2, a linear tangent line $(mx + c)$ is not ideal. Instead, we look for a quadratic tangent curve optimized for x^2, which takes the form

$$g(x) = Ax^2 + B$$

To make $g(x)$ tangent to $f(x)$ at $x = 1$, the curve must match both the slope (derivative) and the value of $f(x)$ at that exact point. The derivative of our tangent curve is

$$g'(x) = 2Ax$$

Setting $g'(1) = f'(1)$ allows us to solve for A, to get

$$2A(1) = \frac{1}{n^2} \Rightarrow A = \frac{1}{2n^2}$$

Setting the value $g(1) = f(1)$ allows us to solve for B, to get

$$A(1)^2 + B = \frac{1}{n}$$

Substituting our newly found value of A yields

$$\frac{1}{2n^2} + B = \frac{1}{n}$$

$$B = \frac{1}{n} - \frac{1}{2n^2} = \frac{2n}{2n^2} - \frac{1}{2n^2} = \frac{2n-1}{2n^2}$$

Now we substitute A and B back into our original assumption $g(x) = Ax^2 + B$ and we obtain

$$g(x) = \left(\frac{1}{2n^2}\right)x^2 + \frac{2n-1}{2n^2} = \frac{x^2+2n-1}{2n^2}$$

This explains the origin of the bound. It is the unique parabolic curve that perfectly touches the graph of $f(x)$ at the equality point $x = 1$.

To verify this bound, we analyze the difference between the two sides:

$$\frac{1}{n+1-x} - \frac{x^2+2n-1}{2n^2} = \frac{2n^2 - (n+1-x)(x^2+2n-1)}{2n^2(n+1-x)}$$

Expanding the numerator results in a polynomial that factors perfectly:

$$\frac{x^3 - (n+1)x^2 + (2n-1)x - (n-1)}{2n^2(n+1-x)} = \frac{(x-1)^2(x-n+1)}{2n^2(n+1-x)}$$

Now we determine the sign of this expression for $x = a_k$:

1. $(x-1)^2 \geq 0$ is always true.
2. Since $\sum_{k=1}^{n} a_k^2 = n$, the maximum value any single variable can take is $a_k \leq \sqrt{n}$.
3. Since $n \geq 3$, we have $a_k \leq \sqrt{n} < n+1$, meaning the denominator $n+1-x > 0$ is positive.
4. Since $n \geq 3$, it follows that $\sqrt{n} \leq n-1$. Therefore, $a_k \leq n-1$, which implies $(x-n+1) \leq 0$.

Since the term $(x-n+1)$ is non-positive and all other components are positive, the entire expression is less than or equal to 0. Thus, our bound holds:

$$\frac{1}{n+1-a_k} \leq \frac{a_k^2+2n-1}{2n^2}$$

Summing this inequality cyclically for all $k = 1, 2, \dots, n$:

$$\sum_{k=1}^{n} \frac{1}{n+1-a_k} \leq \sum_{k=1}^{n} \frac{a_k^2+2n-1}{2n^2}$$

$$\sum_{k=1}^{n} \frac{1}{n+1-a_k} \leq \frac{1}{2n^2}\left(\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n}(2n-1)\right)$$

Substituting the given condition $\sum_{k=1}^{n} a_k^2 = n$ yields

$$\sum_{k=1}^{n} \frac{1}{n+1-a_k} \leq \frac{1}{2n^2}\left(n + n(2n-1)\right)$$

$$\sum_{k=1}^{n} \frac{1}{n+1-a_k} \leq \frac{n+2n^2-n}{2n^2} = \frac{2n^2}{2n^2} = 1$$

Thus, the inequality holds for all $n \geq 1$. Equality is achieved if and only if $a_1 = a_2 = \cdots = a_n = 1$.