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J.3358. *Proposed by Mihály Bencze and Necualai Stanciu, Romania.* Let $a_k > 0$ for $k = 1, 2, \ldots, n$. Prove the following cyclic inequalities

$$\sum_{\text{cyclic}} \frac{a_1^2}{a_1 + a_2} \geq \frac{1}{2} \sum_{k=1}^{n} a_k \geq \sum_{\text{cyclic}} \frac{a_1 a_2^2}{a_1^2 + a_2^2}$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. We split the proof into two parts:

1. The Left-Hand Inequality (LHS). We want to show that

$$\sum_{\text{cyclic}} \frac{a_1^2}{a_1 + a_2} \geq \frac{1}{2} \sum_{k=1}^{n} a_k$$

First, consider the algebraic identity for an individual term in the cycle

$$\frac{a_1^2}{a_1 + a_2} - \frac{a_2^2}{a_1 + a_2} = \frac{a_1^2 - a_2^2}{a_1 + a_2} = \frac{(a_1 - a_2)(a_1 + a_2)}{a_1 + a_2} = a_1 - a_2$$

Summing this identity cyclically over all variables from 1 to n gives

$$\sum_{\text{cyclic}} \frac{a_1^2}{a_1 + a_2} - \sum_{\text{cyclic}} \frac{a_2^2}{a_1 + a_2} = \sum_{\text{cyclic}} (a_1 - a_2)$$

Since the right side is a cyclic telescoping sum, it cancels out completely

$$\sum_{\text{cyclic}} (a_1 - a_2) = (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_n - a_1) = 0$$

This establishes that

$$\sum_{\text{cyclic}} \frac{a_1^2}{a_1 + a_2} = \sum_{\text{cyclic}} \frac{a_2^2}{a_1 + a_2}$$

Now, we add $\displaystyle\sum_{\text{cyclic}} \frac{a_1^2}{a_1 + a_2}$ to itself, substituting the equivalent identity for

$$2 \sum_{\text{cyclic}} \frac{a_1^2}{a_1 + a_2} = \sum_{\text{cyclic}} \frac{a_1^2}{a_1 + a_2} + \sum_{\text{cyclic}} \frac{a_2^2}{a_1 + a_2} = \sum_{\text{cyclic}} \frac{a_1^2 + a_2^2}{a_1 + a_2}$$

Using the standard inequality $2(a_1^2 + a_2^2) \geq (a_1 + a_2)^2$, we have

$$\frac{a_1^2 + a_2^2}{a_1 + a_2} \geq \frac{a_1 + a_2}{2}$$

Summing this cyclically yields

$$2\sum_{\text{cyclic}}\frac{a_1^2}{a_1+a_2} \geq \sum_{\text{cyclic}}\frac{a_1+a_2}{2} = \frac{1}{2}\left(2\sum_{k=1}^{n}a_k\right) = \sum_{k=1}^{n}a_k$$

From which it follows

$$\sum_{\text{cyclic}}\frac{a_1^2}{a_1+a_2} \geq \frac{1}{2}\sum_{k=1}^{n}a_k$$

2. The Right-Hand Inequality (RHS). We want to show that

$$\frac{1}{2}\sum_{k=1}^{n}a_k \geq \sum_{\text{cyclic}}\frac{a_1a_2^2}{a_1^2+a_2^2}$$

By the AM-GM inequality, we know that

$$a_1^2+a_2^2 \geq 2a_1a_2$$

Taking the reciprocal of both sides reverses the inequality

$$\frac{1}{a_1^2+a_2^2} \leq \frac{1}{2a_1a_2}$$

Multiplying by the positive numerator $a_1a_2^2$ gives

$$\frac{a_1a_2^2}{a_1^2+a_2^2} \leq \frac{a_1a_2^2}{2a_1a_2} = \frac{a_2}{2}$$

Adding up this relationship cyclically across all variables yields

$$\sum_{\text{cyclic}}\frac{a_1a_2^2}{a_1^2+a_2^2} \leq \sum_{\text{cyclic}}\frac{a_2}{2} = \frac{1}{2}\sum_{k=1}^{n}a_k$$

Which completes the right-hand inequality. Combining the bounds from Part 1 and Part 2 yields the final desired chain

$$\sum_{\text{cyclic}}\frac{a_1^2}{a_1+a_2} \geq \frac{1}{2}\sum_{k=1}^{n}a_k \geq \sum_{\text{cyclic}}\frac{a_1a_2^2}{a_1^2+a_2^2}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.