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J.3356. *Proposed by Neculai Stanciu, Romania.* If $x_k > 0$ $(k = 1, 2, \ldots, n)$ and $\sum_{k=1}^{n} x_k = 1$, then prove that $\displaystyle\sum_{k=1}^{n} \frac{n^2+1}{n^3(1+x_k^2)} \leq 1.$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. First, we observe that

$$\frac{n^2+1}{n^3} \sum_{k=1}^{n} \frac{1}{1+x_k^2} \leq 1 \iff \sum_{k=1}^{n} \frac{1}{1+x_k^2} \leq \frac{n^3}{n^2+1}$$

We will analyze this inequality using the Tangent Line Method at the symmetric point where $x_k = 1/n$ for all k. Indeed, consider the function $f(x) = \dfrac{1}{1+x^2}$. We find the equation of the tangent line to $f(x)$ at $x_0 = 1/n$. We have

1. Function value at x_0:

$$f\left(\frac{1}{n}\right) = \frac{1}{1+\frac{1}{n^2}} = \frac{n^2}{n^2+1}$$

2. First derivative of $f(x)$:

$$f'(x) = -\frac{2x}{(1+x^2)^2}$$

3. Derivative value at x_0:

$$f'\left(\frac{1}{n}\right) = -\frac{\frac{2}{n}}{\left(1+\frac{1}{n^2}\right)^2} = -\frac{2n^3}{(n^2+1)^2}$$

Then the tangent line $L(x)$ at $x_0 = 1/n$ is given by:

$$L(x) = f\left(\frac{1}{n}\right) + f'\left(\frac{1}{n}\right)\left(x - \frac{1}{n}\right) = \frac{n^2}{n^2+1} - \frac{2n^3}{(n^2+1)^2}\left(x - \frac{1}{n}\right)$$

We want to show that $f(x) \leq L(x)$ for $x \in (0, 1)$. We have

$$L(x) - f(x) = \frac{n^4 + 3n^2 - 2n^3 x}{(n^2+1)^2} - \frac{1}{1+x^2}$$

Combining these over a common denominator gives a numerator of:

$$P(x) = (n^4 + 3n^2 - 2n^3 x)(1 + x^2) - (n^2+1)^2$$

$$P(x) = -2n^3 x^3 + (n^4 + 3n^2)x^2 - 2n^3 x + n^2 - 1$$

Since $x = 1/n$ is the tangent point, $(nx-1)^2$ must be a factor. Factoring $P(x)$, we obtain:

$$P(x) = (nx-1)^2(n^2 - 1 - 2nx)$$

For $f(x) \leq L(x)$ to hold, we need $P(x) \geq 0$, which requires:

$$n^2 - 1 - 2nx \geq 0 \Rightarrow x \leq \frac{n^2 - 1}{2n}$$

If $n \geq 3$, then $\frac{n^2-1}{2n} \geq \frac{8}{6} > 1$. Since $0 < x_k < 1$, the condition $x \leq \frac{n^2-1}{2n}$ is automatically satisfied for all possible values of x_k. Thus, $f(x_k) \leq L(x_k)$ holds for all $n \geq 3$.

Summing $f(x_k) \leq L(x_k)$ from $k = 1$ to n yields

$$\sum_{k=1}^{n} \frac{1}{1 + x_k^2} \leq \sum_{k=1}^{n} \left[\frac{n^2}{n^2 + 1} - \frac{2n^3}{(n^2 + 1)^2} \left(x_k - \frac{1}{n} \right) \right]$$

Using the constrain $\sum_{k=1}^{n} x_k = 1$, we obtain

$$\sum_{k=1}^{n} \frac{1}{1 + x_k^2} \leq \frac{n^3}{n^2 + 1} - \frac{2n^3}{(n^2 + 1)^2} \left(\sum_{k=1}^{n} x_k - n \cdot \frac{1}{n} \right)$$

$$\sum_{k=1}^{n} \frac{1}{1 + x_k^2} \leq \frac{n^3}{n^2 + 1} - \frac{2n^3}{(n^2 + 1)^2}(1 - 1) = \frac{n^3}{n^2 + 1}$$

from which the desired inequality

$$\sum_{k=1}^{n} \frac{n^2 + 1}{n^3(1 + x_k^2)} \leq 1$$

follows.

It remains to analyze the cases $n = 1$ and $n = 2$. The case $n = 1$ trivially holds $(1 = 1)$, and when $n = 2$, the inequality becomes

$$\frac{1}{1 + x_1^2} + \frac{1}{1 + x_2^2} \leq \frac{8}{5}$$

with $x_1 + x_2 = 1$. Letting $x_1 = x$ and $x_2 = 1 - x$, we substitute into the expression:

$$\frac{1}{1 + x^2} + \frac{1}{1 + (1 - x)^2} = \frac{2x^2 - 2x + 3}{x^4 - 2x^3 + 3x^2 - 2x + 2} \leq \frac{8}{5}$$

Cross-multiplying and simplifying, gives

$$5(2x^2 - 2x + 3) \leq 8(x^4 - 2x^3 + 3x^2 - 2x + 2)$$

$$0 \leq 8x^4 - 16x^3 + 14x^2 - 6x + 1 \Leftrightarrow 0 \leq (2x - 1)^2(2x^2 - 2x + 1)$$

Since $(2x - 1)^2 \geq 0$ and $2x^2 - 2x + 1 = x^2 + (1 - x)^2 > 0$, this is always true.

Finally, we conclude that the inequality holds true for all $n \geq 1$. Equality is achieved if and only if $x_1 = x_2 = \ldots = x_n = 1/n$.