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J.3355. *Proposed by Mihály Bencze and Necualai Stanciu, Romania.* Let a, b, c be three positive real numbers. Prove that

$$\frac{a}{2b+7\cdot\sqrt[4]{ab^3}}+\frac{b}{2c+7\cdot\sqrt[4]{bc^3}}+\frac{c}{2a+7\cdot\sqrt[4]{ca^3}}\geq\frac{1}{3}$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. By the AM-GM inequality, we can bound the radical term in the denominator:

$$\sqrt[4]{ab^3}=\sqrt[4]{a\cdot b\cdot b\cdot b}\leq\frac{a+b+b+b}{4}=\frac{a+3b}{4}$$

Substituting this upper bound back into the first denominator yields

$$2b+7\sqrt[4]{ab^3}\leq 2b+7\left(\frac{a+3b}{4}\right)=\frac{8b+7a+21b}{4}=\frac{7a+29b}{4}$$

Since maximizing the denominator minimizes the value of the fraction, we obtain the following inequality

$$\frac{a}{2b+7\sqrt[4]{ab^3}}\geq\frac{a}{\frac{7a+29b}{4}}=\frac{4a}{7a+29b}$$

By applying the same cyclic permutation to the remaining two terms, we get

$$\frac{b}{2c+7\sqrt[4]{bc^3}}\geq\frac{4b}{7b+29c}$$

$$\frac{c}{2a+7\sqrt[4]{ca^3}}\geq\frac{4c}{7c+29a}$$

Adding up these three results together, the problem reduces to prove that

$$\sum_{cyc}\frac{4a}{7a+29b}\geq\frac{1}{3}\Leftrightarrow\sum_{cyc}\frac{a}{7a+29b}\geq\frac{1}{12}$$

To handle the cyclic fractions, we rewrite each term to apply Cauchy-Schwarz inequality:

$$\sum_{cyc}\frac{a}{7a+29b}=\sum_{cyc}\frac{a^2}{7a^2+29ab}$$

Applying CBS gives

$$\sum_{cyc}\frac{a^2}{7a^2+29ab}\geq\frac{(a+b+c)^2}{(7a^2+29ab)+(7b^2+29bc)+(7c^2+29ca)}$$

$$\geq\frac{(a+b+c)^2}{7(a^2+b^2+c^2)+29(ab+bc+ca)}$$

It remains to show that this lower bound is greater than or equal to $\frac{1}{12}$:

$$\frac{(a+b+c)^2}{7(a^2+b^2+c^2)+29(ab+bc+ca)} \geq \frac{1}{12}$$

Expanding the numerator $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$ and cross-multiplying results in

$$12(a^2+b^2+c^2)+24(ab+bc+ca) \geq 7(a^2+b^2+c^2)+29(ab+bc+ca)$$

Subtracting $7(a^2+b^2+c^2)$ and $24(ab+bc+ca)$ from both sides simplifies the expression to

$$a^2+b^2+c^2 \geq ab+bc+ca$$

Since $a^2+b^2+c^2 \geq ab+bc+ca$ is a well-known fundamental inequality that holds true for all real numbers, the original inequality is proven. Equality holds if and only if $a=b=c$.