José Luis **Díaz-Barrero**
Barcelona Mathematical Circle (BMC)
jose.luis.diaz@upc.edu

J.3351. *Proposed by Marin Chirciu, Romania.* In any triangle ABC, let h_a, h_b, h_c denote the altitudes, R the circumradius, and r the inradius. Prove that it holds

$$\sum_{\text{cyc}} \frac{h_a^2}{h_c} \leq \frac{9R}{2}\left(\frac{R}{2r}\right)^3$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Let us break the proof down into three steps: simplifying the cyclic sum, applying known geometric inequalities, and comparing both sides.

1. Expressing the LHS in terms of Altitudes. First, recall that the altitudes of a triangle can be ordered. Without loss of generality, assume the side lengths of the triangle are ordered as $a \leq b \leq c$. This directly implies the altitudes are ordered in the opposite direction:

$$h_a \geq h_b \geq h_c$$

Using the rearrangement inequality on the sequences (h_a, h_b, h_c) and $\left(\frac{1}{h_a}, \frac{1}{h_b}, \frac{1}{h_c}\right)$, we can establish a clean upper bound for our cyclic sum:

$$\sum_{\text{cyc}} \frac{h_a^2}{h_c} = \frac{h_a^2}{h_c} + \frac{h_b^2}{h_a} + \frac{h_c^2}{h_b} \leq h_a + h_b + h_c$$

Thus, the LHS of our inequality is bounded by the sum of the altitudes

$$\text{LHS} \leq h_a + h_b + h_c$$

2. Bounding the Sum of Altitudes. Next, we relate the sum of the altitudes to the inradius r and the circumradius R. We know that in any triangle, the inradius r relates to the altitudes via:

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

Using the relationship between the arithmetic mean (AM) and the harmonic mean (HM) for the altitudes

$$\frac{h_a + h_b + h_c}{3} \geq \frac{3}{\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}} = 3r$$

we get the bound $h_a + h_b + h_c \geq 9r$. To find an upper bound in terms of the circumradius R, we use the well-known geometric identity (it will be proven later on):

$$h_a + h_b + h_c \leq \frac{9R}{2}$$

This inequality is a standard result (equality holds if and only if the triangle is equilateral, where $h_a = h_b = h_c = \frac{3R}{2}$).
Combining our results from Step 1 and Step 2, we have

$$\text{LHS} \le h_a + h_b + h_c \le \frac{9R}{2}$$

3. Comparison of LHS and RHS using Euler's Inequality. Now, let us examine the right-hand side (RHS) of our target inequality:

$$\text{RHS} = \frac{9R}{2}\left(\frac{R}{2r}\right)^3$$

By Euler's Inequality, for any triangle, the circumradius is at least twice the inradius:

$$R \ge 2r \Rightarrow \frac{R}{2r} \ge 1$$

Since $\dfrac{R}{2r} \ge 1$, cubing it keeps it greater than or equal to 1. That is,

$$\left(\frac{R}{2r}\right)^3 \ge 1$$

We can now chain our inequalities together:

$$\text{LHS} \le h_a + h_b + h_c \le \frac{9R}{2} \le \frac{9R}{2}\left(\frac{R}{2r}\right)^3 = \text{RHS}$$

This proves that

$$\sum_{\text{cyc}} \frac{h_a^2}{h_c} \le \frac{9R}{2}\left(\frac{R}{2r}\right)^3$$

Equality holds if and only if both:

1. $h_a = h_b = h_c$ (from the rearrangement and AM-HM steps), and
2. $R = 2r$ (from Euler's Inequality).

Both conditions are met simultaneously if and only if the triangle is equilateral. For all other triangles, as the shape deviates from equilateral, the ratio $\frac{R}{2r}$ grows larger than 1, making the RHS strictly larger than the LHS.

Finally, we prove the identity previously claimed:

Lemma. *In any triangle ABC with circumradius R and altitudes h_a, h_b, h_c, the following inequality holds*

$$h_a + h_b + h_c \le \frac{9R}{2}$$

Equality holds if and only if the triangle is equilateral.

Proof. Let us prove this identity in three clear steps using basic trigonometry.

1. Expressing Altitudes in Terms of Angles. Recall from basic right-triangle trigonometry that in $\triangle ABC$, the altitude h_a can be written using the sides b or

c as $h_a = c \sin B$. By the Law of Sines, we know that $c = 2R \sin C$. Substituting this into our formula for h_a gives

$$h_a = 2R \sin B \sin C$$

By symmetry, we can write all three altitudes in terms of the circumradius R and the angles of the triangle

$$h_a = 2R \sin B \sin C, \quad h_b = 2R \sin C \sin A, \quad h_c = 2R \sin A \sin B$$

Summing these up, we get

$$h_a + h_b + h_c = 2R(\sin A \sin B + \sin B \sin C + \sin C \sin A)$$

2. Using an Algebraic Inequality. An elementary algebraic identity states that for any three real numbers x, y, z, it holds

$$xy + yz + zx \leq \frac{(x+y+z)^2}{3}$$

Letting $x = \sin A$, $y = \sin B$, and $z = \sin C$, we can apply this directly to our sum of sine products, to obtain

$$\sin A \sin B + \sin B \sin C + \sin C \sin A \leq \frac{(\sin A + \sin B + \sin C)^2}{3}$$

3. Bounding the Sum of Sines (Jensen's Inequality). Since the function $f(x) = \sin x$ is strictly concave on the interval $(0, \pi)$, we can apply Jensen's Inequality for the three angles of a triangle ($A + B + C = \pi$). We have

$$\frac{\sin A + \sin B + \sin C}{3} \leq \sin\left(\frac{A+B+C}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

form which it follows that

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$$

Now, we square both sides of this bound and we get

$$(\sin A + \sin B + \sin C)^2 \leq \left(\frac{3\sqrt{3}}{2}\right)^2 = \frac{27}{4}$$

Substituting this result back into our inequality from Step 2 yields

$$\sin A \sin B + \sin B \sin C + \sin C \sin A \leq \frac{\frac{27}{4}}{3} = \frac{9}{4}$$

Finally, putting this back into our original formula for the sum of the altitudes from Step 1, we obtain

$$h_a + h_b + h_c = 2R(\sin A \sin B + \sin B \sin C + \sin C \sin A) \leq 2R\left(\frac{9}{4}\right) = \frac{9R}{2}$$

Thus, we have shown that

$$h_a + h_b + h_c \leq \frac{9R}{2}$$

Equality holds if and only if $\sin A = \sin B = \sin C$, meaning $A = B = C = 60^\circ$, which defines an equilateral triangle.