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J.3335. *Proposed by Marin Chirciu, Romania.* Let $a, b, c > 0$ such that $abc = 1$. Prove that

$$\frac{1}{2}(a+b)^3 + \frac{b^2+c^2}{a^2} + \frac{c^2+a^2}{b^2} \geq 8.$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. First, let us analyze and simplify the last two terms of the inequality using the AM-GM inequality

$$\frac{b^2+c^2}{a^2} + \frac{c^2+a^2}{b^2} = \frac{b^2}{a^2} + \frac{c^2}{a^2} + \frac{c^2}{b^2} + \frac{a^2}{b^2}$$

Using AM-GM on the individual terms yields

$$\frac{b^2}{a^2} + \frac{a^2}{b^2} \geq 2\sqrt{\frac{b^2}{a^2} \cdot \frac{a^2}{b^2}} = 2$$

This simplifies the expression to

$$\frac{b^2+c^2}{a^2} + \frac{c^2+a^2}{b^2} \geq 2 + c^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

Applying AM-GM again inside the parenthesis, we get

$$\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{2}{ab}$$

Since $abc = 1$, we have $\dfrac{1}{ab} = c$. Substituting this, we obtain

$$c^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) \geq c^2\left(\frac{2}{ab}\right) = 2c^3$$

Therefore, the lower bound for the last two terms is

$$\frac{b^2+c^2}{a^2} + \frac{c^2+a^2}{b^2} \geq 2 + 2c^3$$

Now, let us examine the first term of our original inequality: $\dfrac{1}{2}(a+b)^3$. Using the AM-GM inequality on a and b yields $a + b \geq 2\sqrt{ab}$. Cubing both sides we obtain

$$(a+b)^3 \geq 8(ab)^{3/2} \Leftrightarrow \frac{1}{2}(a+b)^3 \geq 4(ab)^{3/2}$$

Using the constraint $abc = 1$, we can rewrite ab as $\dfrac{1}{c}$, and we get

$$\frac{1}{2}(a+b)^3 \geq 4\left(\frac{1}{c}\right)^{3/2} = \frac{4}{c\sqrt{c}}$$

Combining our two bounds, the LHS of the original inequality satisfies

$$\text{LHS} \geq \frac{4}{c\sqrt{c}} + 2 + 2c^3$$

Let $x = c^{3/2} > 0$. Then $c^3 = x^2$ and $c\sqrt{c} = c^{3/2} = x$. We can rewrite the expression in terms of x as

$$\text{LHS} \geq \frac{4}{x} + 2x^2 + 2$$

To find the minimum of $f(x) = \dfrac{4}{x} + 2x^2 + 2$ for $x > 0$, we can apply the AM-GM inequality by splitting the $\dfrac{4}{x}$ term, to obtain

$$\frac{4}{x} + 2x^2 = \frac{2}{x} + \frac{2}{x} + 2x^2 \geq 3\sqrt[3]{\frac{2}{x} \cdot \frac{2}{x} \cdot 2x^2} = 3\sqrt[3]{8} = 6$$

Thus:

$$\text{LHS} \geq 6 + 2 = 8$$

Equality holds if and only if $a = b = c = 1$. This completes the proof.