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J.3326. *Proposed by Dumitru Bătineţu-Giurgiu, and Claudia Nănuţi, Romania.* Let $ABCD$ be a convex quadrilateral with side lengths $a = AB$, $b = BC$, $c = CD$, and $d = DA$, and semiperimeter s. Prove that

$$\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} + \frac{d}{s-d} \geq 4$$

Solution by José Luis Díaz Barrero, Barcelona, Spain. Let us rewrite each term in the sum. Adding and subtracting s in the numerator of each fraction yields

$$\frac{a}{s-a} = \frac{a-s+s}{s-a} = \frac{-(s-a)+s}{s-a} = -1 + \frac{s}{s-a}$$

Applying this algebraic identity to all four terms, the inequality becomes:

$$\left(-1 + \frac{s}{s-a}\right) + \left(-1 + \frac{s}{s-b}\right) + \left(-1 + \frac{s}{s-c}\right) + \left(-1 + \frac{s}{s-d}\right) \geq 4,$$

$$s\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{1}{s-d}\right) - 4 \geq 4,$$

or

$$s\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{1}{s-d}\right) \geq 8$$

On account of CBS (or AM-HM) inequality, for any positive real numbers x_1, x_2, x_3, x_4, we have

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} \geq \frac{16}{x_1 + x_2 + x_3 + x_4}$$

Letting $x_1 = s - a$, $x_2 = s - b$, $x_3 = s - c$, and $x_4 = s - d$, we have

$$(s-a) + (s-b) + (s-c) + (s-d) = 4s - (a+b+c+d)$$

Since s is the semiperimeter, $a + b + c + d = 2s$. Substituting this in gives $4s - 2s = 2s$. Now, applying the previous inequality yields

$$\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{1}{s-d} \geq \frac{16}{2s} = \frac{8}{s}$$

Multiplying both sides of this inequality by s (which is strictly positive)yields

$$s\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{1}{s-d}\right) \geq 8 \Leftrightarrow s\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{1}{s-d}\right) - 4 \geq 4$$

which is equivalent to

$$\frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} + \frac{d}{s-d} \geq 4$$

Equality holds if and only if $s - a = s - b = s - c = s - d$, which implies $a = b = c = d$ (the quadrilateral is a rhombus). This completes the proof.