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J.3318. *Proposed by Dumitru Bătineţu-Giurgiu, Romania.* In any triangle ABC with area F, the following inequality holds:

$$\frac{a^5}{b+c}+\frac{b^5}{c+a}+\frac{c^5}{a+b}\geq 8F^2.$$

Solution by José Luis Díaz-Barrero, Barcelona, Spain. By applying CBS , we can find a lower bound for the left-hand side in terms of the sides of the triangle. We claim that

$$\frac{a^5}{b+c}+\frac{b^5}{c+a}+\frac{c^5}{a+b}\geq \frac{(a^2+b^2+c^2)^3}{2(a+b+c)^2}$$

Indeed, to establish the preceding inequality we apply the Cauchy-Schwarz inequality in two distinct stages. First, we apply the standard Cauchy-Schwarz inequality to the sequences of terms. Consider the product:

$$\frac{a^5}{b+c}+\frac{b^5}{c+a}+\frac{c^5}{a+b}=\frac{a^6}{a(b+c)}+\frac{b^6}{b(c+a)}+\frac{c^6}{c(a+b)}\geq\frac{\left(a^3+b^3+c^3\right)^2}{a(b+c)+b(c+a)+c(a+b)}$$

Expanding and simplifying the last denominator yields

$$a(b+c)+b(c+a)+c(a+b)=2(ab+bc+ca)$$

and we get our first intermediate bound:

$$\frac{a^5}{b+c}+\frac{b^5}{c+a}+\frac{c^5}{a+b}\geq\frac{(a^3+b^3+c^3)^2}{2(ab+bc+ca)}\qquad(1)$$

Second Application of Cauchy-Schwarz. Next, we relate the cubic sum to the quadratic sum. Applying Cauchy-Schwarz to the vectors $(\sqrt{a^3},\sqrt{b^3},\sqrt{c^3})$ and $(\sqrt{a},\sqrt{b},\sqrt{c})$:

$$\left(a^3+b^3+c^3\right)(a+b+c)\geq\left(a^2+b^2+c^2\right)^2$$

from which it follows

$$a^3+b^3+c^3\geq\frac{(a^2+b^2+c^2)^2}{a+b+c}$$

Squaring both sides yields

$$\left(a^3+b^3+c^3\right)^2\geq\frac{(a^2+b^2+c^2)^4}{(a+b+c)^2}\qquad(2)$$

Substituting Inequality (2) into the numerator of Inequality (1) gives:

$$\frac{a^5}{b+c}+\frac{b^5}{c+a}+\frac{c^5}{a+b}\geq\frac{(a^2+b^2+c^2)^4}{2(ab+bc+ca)(a+b+c)^2}$$

Using the well-known algebraic identity $a^2+b^2+c^2 \geq ab+bc+ca$, we can replace the term in the denominator. Because we replace it with a larger or equal value, the overall fraction becomes smaller or equal, preserving the inequality:

$$\frac{a^5}{b+c}+\frac{b^5}{c+a}+\frac{c^5}{a+b} \geq \frac{(a^2+b^2+c^2)^4}{2(a^2+b^2+c^2)(a+b+c)^2}$$

Canceling out one factor of $(a^2+b^2+c^2)$ from both the numerator and the denominator yields the claimed inequality

$$\frac{a^5}{b+c}+\frac{b^5}{c+a}+\frac{c^5}{a+b} \geq \frac{(a^2+b^2+c^2)^3}{2(a+b+c)^2}$$

To relate the side lengths to the area of the triangle, we employ two classical well-known geometric inequalities:

1. Weitzenböck's Inequality: For any triangle, the sum of the squares of the sides is bounded below by the area:

$$a^2+b^2+c^2 \geq 4\sqrt{3}F$$

2. Isoperimetric Inequality: For any triangle, the perimeter is bounded below by the area:

$$(a+b+c)^2 \geq 12\sqrt{3}F$$

Substituting these two geometric bounds into our algebraic previous result yields

$$\frac{(a^2+b^2+c^2)^3}{2(a+b+c)^2} \geq \frac{(4\sqrt{3}F)^3}{2(12\sqrt{3}F)}$$

Now, we simplify the right-hand side. Expanding the numerator and denominator gives

$$\frac{64 \cdot 3\sqrt{3} \cdot F^3}{24\sqrt{3}F} = \frac{192\sqrt{3}F^3}{24\sqrt{3}F} = 8F^2$$

Thus, by chaining the inequalities, we establish

$$\frac{a^5}{b+c}+\frac{b^5}{c+a}+\frac{c^5}{a+b} \geq 8F^2$$

Since equality in all used inequalities requires $a=b=c$, then inequality holds, with equality if and only if $\triangle ABC$ is equilateral.